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Spectral stability estimates for the eigenfunctions of second order elliptic operators

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Introduction

The problem of spectral stability is one of the important problems in the theory of partial differential operators, which apart from theoretical interest, is also useful in applications, first of all to numerical methods related to computing eigenvalues and eigenfunctions.

First results on this problem, related to continuous dependence of the eigenvalues upon domain variation, could be found in the famous monograph of Courant and Hilbert [16, Ch.VI. §2.6]. Currently, there is vast literature concerning domain perturbation problems, see for instance the extensive monograph by Henry [33], the survey papers by Hale [32].

The thesis is dedicated to the problem of spectral stability of a non-negative self-adjoint elliptic operator

$$Su = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j},$$

on an open set $\Omega \subset \mathbb{R}^n$, where a_{ij}, $i,j = 1,\dots,n$, are Lipschitz continuous functions such that the matrix (a_{ij}) is Hermitian and for some $\theta > 0$

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \geq \theta|\xi|^2$$

for all $\xi \in \mathbb{C}^n$ and $x \in \Omega$, subject to homogeneous Dirichlet boundary conditions.

It is assumed that this operator has compact resolvent, hence its spectrum consists only of eigenvalues of finite multiplicity

$$0 < \lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_k[\Omega] \leq \cdots .$$

Here each eigenvalue is repeated as many times as its multiplicity. Also

$$\lim_{k\to\infty} \lambda_k[\Omega] = \infty.$$

The problem of continuous dependence of eigenvalues $\lambda_k[\Omega]$ upon domain perturbation and the problem of obtaining estimates for the variation of the eigenvalues for elliptic operators will homogeneous Dirichlet boundary conditions has been studied in detail in [11], [12], [13], [14].

We quote some of the most recent results in this direction. In [14] an atlas $\mathcal{A}$ is fixed and perturbations of the domain only within $C(\mathcal{A})$, (see § 2.4) are allowed, thus the variation of the eigenvalues is estimated from above by the so-called atlas distance of the domains up to a multiplicative positive constant that depends only on the operator itself, the index k of the eigenvalue in the above mentioned enumeration and the atlas. The same kind of estimate is achieved in [14] with the modulus of continuity $\omega(\cdot)$, that determines the class $C^{\omega(\cdot)}(\mathcal{A})$ (see again § 2.4) in which the domains are assumed to vary, evaluated at the Haussdorff-Pompeiu lower deviation of the domains. As a corollary, if the domains, say Ω_1, Ω_2, are assumed with Lipschitz continuous boundaries, and $(\Omega_1)_\varepsilon \subset \Omega_2 \subset (\Omega_1)^\varepsilon$ or $(\Omega_2)_\varepsilon \subset \Omega_1 \subset (\Omega_2)^\varepsilon$, then

$$|\lambda_k[\Omega_1] - \lambda_k[\Omega_2]| \leq c_k\varepsilon$$

if ε is sufficiently small. The results cited above hold for elliptic operators of any order. For operators of second, in [13] the variation of the eigenvalues has also been estimated in terms of the Lebesgue measure of the symmetric difference of the domains.

Another important problem in this area is the problem of the continuous dependence upon domain perturbation of the eigenfunctions. However, this problem is much less investigated, see [45], [46].

In the recent paper [6], in order to compare the operators S_1 and S_2 on open sets Ω_1, Ω_2 respectively, the actual comparison is made for the so-called pull-back operators. H_1, H_2 to a fixed open set Ω such that $\Omega_1 = \psi_1(\Omega)$ and $\Omega_2 = \psi_2(\Omega)$, and in fact dependence of the eigenfunctions $\varphi_n[\psi(\Omega)]$ on transformations ψ from a certain class is investigated. Let

$$\delta_p(\psi_1, \psi_2) = \|\nabla\psi_1 - \nabla\psi_2\|_{L^p(\Omega)} + \|A \circ \psi_1 - A \circ \psi_2\|_{L^p(\Omega)},$$

where $A = (a_{ij})_{i,j=1,\dots,n}$ is the matrix of the coefficients. The main result of [6] for the eigenfunctions has the following form. Under certain assumptions on the coefficients for each $k \in \mathbb{N}$ there exist $c_k, \varepsilon_k > 0$ such that if λ is an eigenvalue of S_1 on Ω_1 of multiplicity m and k is such that $\lambda = \lambda_k[\Omega_1] = \cdots = \lambda_{k+m-1}[\Omega_1]$, and $\varphi_k[\Omega_2], \dots, \varphi_{k+m-1}[\Omega_2]$ are orthonormal eigenfunctions of S_2 on Ω_2 corresponding to the eigenvalues $\lambda_k[\Omega_2] \le \cdots \le \lambda_{k+m-1}[\Omega_1]$, then for certain $p > 2$ there exist $\varphi_k[\Omega_1], \dots, \varphi_{k+m-1}[\Omega_1]$, orthonormal eigenfunctions of S_1 corresponding to the eigenvalue λ such that

$$\begin{aligned}\|\varphi_{k+i}[\Omega_1] - \varphi_{k+i}[\Omega_2]\|_{L^2(\Omega_1\cup\Omega_2)} \le \\ c_k \left(\delta_p(\psi_1, \psi_2) + \|\varphi_{k+i}[\Omega_1] \circ \psi_1 - \varphi_{k+i}[\Omega_1] \circ \psi_2\|_{L^2(\Omega)} + \right. \\ \left. \|\varphi_{k+i}[\Omega_2] \circ \psi_1 - \varphi_{k+i}[\Omega_2] \circ \psi_2\|_{L^2(\Omega)}\right),\end{aligned}$$

for $i = 0, \dots, m-1$, if $\delta_p(\psi_1, \psi_2) < \varepsilon_k$. (It is assumed that the functions $\varphi_{k+i}[\Omega_1]$, $\varphi_{k+i}[\Omega_2]$ are extended by zero outside Ω_1, Ω_2 respectively.)

For some particular cases more explicit estimates for $\|\varphi_{k+i}[\Omega_1]-\varphi_{k+i}[\Omega_2]\|_{L^2(\Omega_1\cup\Omega_2)}$ are deduced from the above estimate: namely for the case of local perturbations when Ω_2 differs from Ω_1 only in a small neighborhood of a point of the boundary of Ω_1, and for the case of a global normal perturbation when Ω_2 is obtained from Ω_1 by a shift along the normals depending on a parameter $t > 0$. In both cases, under the appropriate assumptions on Ω_1 and Ω_2 from the above estimate the following estimate via the Lebesgue measure of the symmetric difference of Ω_1 and Ω_2 is deduced:

$$\|\varphi_i[\Omega_1] - \varphi_i[\Omega_2]\|_{L^2(\Omega_1\cup\Omega_2)} \le c_k \mathrm{meas}\,(\Omega_1 \triangle \Omega_2)^\gamma,$$

where $\gamma > 0$ is any number such that $\gamma < 1/n$.

The main aim of the thesis is obtaining estimates for generic open sets Ω_1, Ω_2 and in this case a different approach to the problem is required.

We apply the approach based on the one in the monograph of Kato [36] for the case of operators defined on the same domain of the space, in which case dependence on the coefficients of the differential operator can be investigated. In [36] this is done by applying the notion of gap between subspaces and between operators: given two subspaces M, N of a normed space X, the *gap between M and N* is defined as

$$\delta(M, N) = \sup_{\substack{u \in M \\ \|u\| = 1}} \inf_{v \in N} \|u - v\|;$$

and given two linear operators S, T acting between the normed spaces X, Y the *gap between S and T* is

$$\delta(S,T) = \delta(G(S), G(T)),$$

where $G(S)$, and $G(T)$ are the graphs of S and T respectively. Often a version of *symmetrized gap* is needed, namely

$$\hat{\delta}(M,N) = \max\{\delta(M,N),\, \delta(N,M)\},$$

and the corresponding one for the operators,

$$\hat{\delta}(S,T) = \hat{\delta}(G(S), G(T)) = \max\{\delta(S,T),\, \delta(T,S)\}.$$

Sometimes, estimates are obtained even in terms of what here is called *minimal gap*:

$$\delta_{\min}(S,T) = \min\{\delta(S,T),\, \delta(T,S)\} = \delta_{\min}(G(S), G(T)).$$

Actually, all of these notions of *gap* coincide in the case of self-adjoint operators.

In order to modify this approach for the domain perturbation problem, first of all, it is required to define the gap between the operators defined on different domains. In the thesis this is done in the following way. Let $\mathit{\Gamma}$ be a class of bounded open sets of $\mathbb{R}^n$ that satisfy certain conditions, like regularity properties for the boundaries. Moreover, we also require that for any pair of open sets $\Omega_1,\ \Omega_2 \in \mathit{\Gamma}$ such that $\Omega_1 \subset \Omega_2$, then $\Omega_2 \setminus \overline{\Omega_1} \in \mathit{\Gamma}$. We assume that on each open set $\Omega \in \mathit{\Gamma}$ an operator

$$S_\Omega : \mathrm{D}(S_\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

is defined.

Now let $S_{\Omega_1},\ S_{\Omega_2}$ be two operators that are to be compared, defined on $\Omega_1,\ \Omega_2 \in \mathit{\Gamma}$ and let $\Omega_1 \subset \Omega_2$. The canonical identification

$$L^2(\Omega_2) = L^2(\Omega_1) \oplus L^2(\Omega_2 \setminus \Omega_1)$$

permits to consider the *direct sum* operator

$$S_{\Omega_1} \oplus S_{\Omega_2\setminus\overline{\Omega_1}} : \mathrm{D}(S_{\Omega_1} \oplus S_{\Omega_2\setminus\overline{\Omega_1}}) \subset L^2(\Omega_2) \to L^2(\Omega_2).$$

Then we define the *gap between S_{Ω_1} and S_{Ω_2}* by

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \delta(S_{\Omega_1} \oplus S_{\Omega_2\setminus\overline{\Omega_1}},\, S_{\Omega_2}),$$

where the gap in the right side is taken in the space $L^2(\Omega_2)$.

The so-called *spectral subspaces of an operator*, that is, subspaces whose elements are finite sums of eigenfunctions (zero included)–the elements of these spaces here are called *spectral functions*–are introduced. More precisely, with reference to the operator S_Ω let $k,\ m \in \mathbb{N}$ be such that

$$\lambda_{k-1}[\Omega] < \lambda_k[\Omega] \leq \cdots \leq \lambda_{k+m-1}[\Omega] < \lambda_{k+m}[\Omega]$$

(with the understanding that if $k = 1$ the first inequality is not present in the above chain of inequalities), the *spectral subspace* $\mathrm{N}_{k,m}[\Omega]$ is defined as the set of all the linear combinations of the eigenfunctions corresponding to the eigenvalues $\lambda_k[\Omega] \le \cdots \le \lambda_{k+m-1}[\Omega]$.

The main results obtained in the thesis are the following. Let the dimension n of the Euclidean space $\mathbb{R}^n$ assume only the values $1, 2, 3$ and let

$$\gamma = \begin{cases} 1 & \text{if } n = 1, \\ \text{any number in } (0,1) & \text{if } n = 2, \\ \frac{1}{2} & \text{if } n = 3. \end{cases}$$

Let $\mathit{\Gamma}$ be a class of bounded open sets of $\mathbb{R}^n$ whose boundaries satisfy certain regularity properties ($C^{1,r}$-regularity for some $\frac{1}{2} < r < 1$ is sufficient); then we prove that for each $\Omega_2 \in \mathit{\Gamma}$ there exists $M > 0$ such that

$$\hat{\delta}(S_{\Omega_1}, S_{\Omega_2}) \le M\varepsilon^\gamma$$

for all $\varepsilon > 0$ and for all $\Omega_1 \in \mathit{\Gamma}$ such that $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$ and Ω_1, $\Omega_2 \setminus \overline{\Omega_1}$ satisfy the exterior sphere condition.

Further, for each $\Omega_2 \in \mathit{\Gamma}$ and $k, m \in \mathbb{N}$ for which $\lambda_k[\Omega_2]$ is an eigenvalue of multiplicity m for S_{Ω_2}, there exist $M_k, \varepsilon_k > 0$ such that

$$\hat{\delta}(\mathrm{N}_{k,m}[\Omega_1],\ \mathrm{N}_{k,m}[\Omega_2]) \le M_k\varepsilon^\gamma$$

for all $0 < \varepsilon < \varepsilon_0$ and for all $\Omega_1 \in \mathit{\Gamma}$ such that $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$ and Ω_1, $\Omega_2 \setminus \overline{\Omega_1}$ satisfy the exterior sphere condition.

Finally, the last result has an immediate corollary on eigenfunctions in the same fashion as the one cited from [6]. Namely, for each Ω_2, k, m as above there exist $c_k, \varepsilon_k > 0$ such that, if $\varphi_k[\Omega_1], \ldots, \varphi_{k+m-1}[\Omega_1]$ are orthonormal eigenfunctions of S_{Ω_1} corresponding to the eigenvalues $\lambda_k[\Omega_1] \le \cdots \le \lambda_{k+m-1}[\Omega_1]$, then there exist orthonormal eigenfunctions $\varphi_k[\Omega_2], \ldots, \varphi_{k+m-1}[\Omega_2]$ of S_{Ω_2} corresponding to the eigenvalue $\lambda_k[\Omega_2]$ such that

$$\|\varphi_{k+i}[\Omega_1] - \varphi_{k+i}[\Omega_2]\|_{L^2(\Omega_2)} \le c_k\varepsilon^\gamma,$$

$i = 0, \ldots, m-1$, for all $0 < \varepsilon < \varepsilon_0$ and $\Omega_2 \in \mathit{\Gamma}$ such that $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$ and Ω_1, $\Omega_2 \setminus \overline{\Omega_1}$ satisfy the exterior sphere condition.

The thesis is organized in the following way: In §1 is discussed the notion of gap between subspaces and between operators. Mostly we expose known results (on the boundedness and invertibility stability of operators) in the spirit of Kato's monograph [36], but the definition of gap is given in more general terms (Kato in [36] limits himself to closed subspaces and closed operators) and some of the results and proofs are improved. Moreover, in §1.3 various spectral stability estimates are obtained.

For the convenience of the reader, in §2 are collected some facts about self-adjoint operators, the main focus being on variationally defined non-negative self-adjoint operators, that is, operators defined by the means of closed non-negative forms, and Friedrich's extensions.

§3 is concerned with the problem of how to extend the notion of gap to operators defined on different open sets. The main definition adapted here is given in §3.2. Other alternatives are taken into consideration and their shortcomings are briefly discussed.

In §4 the gap, as defined in §3.2, between Dirichlet elliptic operators on different open sets is estimated in terms of the vicinity of the open sets, as described above, and finally, this estimate is employed in §5 to determine spectral stability estimates about the relative eigenfunctions.

The techniques adopted here to estimate the gap, unfortunately, impose a bound on the dimension n of the Euclidean space ($n \leq 3$). The feeling is that this limitation is not substantial but only a technicality. Thus, work remains to be done to discern whether this fact is true or not. Also the problems of the possibility of giving sharper estimates for the gap remains open.

1 Gaps between subspaces and operators

1.1 Gap between subspaces

We begin by introducing some notations. We shall use letters $X, Y, Z, \dots,$ to indicate normed (most often, Banach or Hilbert) spaces. We will denote by $\mathcal{L}(X, Y)$ the set of all (partially defined) linear operators $T : \mathrm{D}(T) \subset X \to Y$, where $\mathrm{D}(T)$ is a linear subspace of X called *the domain of* T. We set also $\mathcal{L}(X) := \mathcal{L}(X, X)$. Furthermore, recall that a linear operator $T : \mathrm{D}(T) \subset X \to Y$ is said to be *closed* if its graph, that is, the set

$$G(T) = \{(u, Tu) : u \in \mathrm{D}(T)\},$$

is closed in[1] $X \times Y$. The operator T is said to be *bounded on* $\mathrm{D}(T)$ if

$$\|T\| := \|T\|_{\mathrm{D}(T) \to Y} := \sup_{\substack{u \in \mathrm{D}(T) \\ u \neq 0}} \frac{\|Tu\|_Y}{\|u\|_X} < \infty;$$

if moreover, $\mathrm{D}(T) = X$ then T is simply said to be *bounded*. We shall denote with $\mathcal{C}(X, Y)$, $\widetilde{\mathcal{B}}(X, Y)$ and $\mathcal{B}(X, Y)$ respectively, in order, the set of all closed, bounded on the corresponding domains and bounded operators from X to Y. In the case $Y = X$, $\mathcal{C}(X, X)$, $\widetilde{\mathcal{B}}(X, Y)$ and $\mathcal{B}(X, X)$ will be denoted simply by $\mathcal{C}(X)$, $\widetilde{\mathcal{B}}(X)$ and $\mathcal{B}(X)$ respectively.

Since we are going to deal with perturbation problems related to linear operators, it is necessary to clarify what is meant by "small" perturbations. This can be done in a most natural way by introducing a metric in $\mathcal{C}(X, Y)$ (e.g., see Kato in [36, Ch.IV]). More generally, we can introduce a pseudometric in $\mathcal{L}(X, Y)$ which induces the foresaid metric in $\mathcal{C}(X, Y)$. As we will see in a while, if $S, T \in \mathcal{L}(X, Y)$, the "distance" between S and T will be measured in terms of the "aperture" or "gap" between the linear subspaces $G(S)$ and $G(T)$ of the product normed space $X \times Y$. Therefore we are lead to consider how to measure the gap between two linear subspaces of a Banach space which is done in the following definition.

Definition 1.1. *The* gap *between two generic linear subspaces M and N of a normed space Z is defined by the following formula:*

$$\delta(M, N) = \sup_{\substack{u \in M \\ \|u\| = 1}} \operatorname{dist}(u, N). \tag{1.1}$$

Remark 1.1. (1.1) *has no meaning if $M = 0$; in this case we define $\delta(0, N) = 0$ for any N. On the other hand $\delta(M, 0) = 1$ if $M \neq 0$ as is seen from the definition.*

$\delta(M, N)$ can be characterized as the smallest number δ such that

$$\operatorname{dist}(u, N) \leq \delta \|u\| \tag{1.2}$$

for all $u \in M$.

[1] The *Cartesian product space* $X \times Y$ (or alternatively called the *direct sum space* and denoted by $X \dot{+} Y$) is recalled briefly at the beginning of the next subsection.

Remark 1.2. *Since for any $c \in \mathbb{R}$ $\operatorname{dist}(cu, N) = |c|\operatorname{dist}(u, N)$, it follows that*

$$\delta(M, N) = \sup_{u \in M,\, \|u\| \leq 1} \operatorname{dist}(u, N). \tag{1.3}$$

We have also that for any real number $c > 0$

$$\sup_{u \in M,\, \|u\| \leq c} \operatorname{dist}(u, N) = c\delta(M, N). \tag{1.4}$$

As it is seen immediately from this definition, the gap $\delta(M, N)$ is not symmetric in M and N. However, sometimes a symmetric version of the gap is useful, hence the following definition is given.

Definition 1.2. *If M and N are linear subspaces of a normed space Z, the* symmetric gap *between M and N is defined by*

$$\hat{\delta}(M, N) = \max[\delta(M, N), \delta(N, M)]. \tag{1.5}$$

Examples 1.1. *1) Let $Z = \mathbb{R}^2$ or $Z = \mathbb{R}^3$ and let M and N be two straight lines passing through the origin such that the angle between them is equal to α. Then $\delta(M, N) = \delta(N, M) = \sin\alpha$.*

2) Let $Z = \mathbb{R}^3$ and let M and N be two planes passing through the origin such that the angle between them is equal to α. Then again $\delta(M, N) = \delta(N, M) = \sin\alpha$.

3) Let $Z = \mathbb{R}^3$, M be a straight line passing through the origin and N be a plane passing through the origin such that the angle between them is equal to α. Then $\delta(M, N) = \sin\alpha$ (in particular, if $M \subset N$ then $\delta(M, N) = 0$), but $\delta(N, M) = 1$.

The following properties of the gap follow directly from the definition.

Proposition 1.1. *Let M and N be two linear subspaces of a normed space Z. Then we have:*

$$\delta(M, N) = \delta(\overline{M}, \overline{N}), \qquad \hat{\delta}(M, N) = \hat{\delta}(\overline{M}, \overline{N}); \tag{1.6}$$

$$\delta(M, N) = 0 \qquad \textit{if, and only if,} \qquad \overline{M} \subset \overline{N}; \tag{1.7}$$

$$\hat{\delta}(M, N) = 0 \qquad \textit{if, and only if,} \qquad \overline{M} = \overline{N}; \tag{1.8}$$

$$\hat{\delta}(M, N) = \hat{\delta}(N, M); \tag{1.9}$$

$$0 \leq \delta(M, N) \leq 1 \qquad 0 \leq \hat{\delta}(M, N) \leq 1. \tag{1.10}$$

If Z is a normed vector space, the symbol Z^* will be used to denote the adjoint (or dual) space of Z. With the usual definitions of addition, multiplication by scalars and norm Z^* becomes a Banach space. If $x \in Z$ and $x^* \in Z^*$, then $(x, x^*)_Z$ will denote the value of the linear form x^* at the vector x, but the subscript will be omitted if no ambiguity arises. The same notation will be used to indicate the inner products in a Hilbert space since, in this case, Z can be identified with Z^*; actually $(x, x^*)_Z$ is just an extension of the notion of inner product to more general spaces.

The notion of the annihilator of a set will also be needed, hence we recall it briefly. Let Z be a normed vector space and let $E \subset Z$, $F \subset Z^*$. The *annihilator* E^0 of E in Z^* is defined by

$$E^0 = \{x^* \in Z^* \; : \; (x, x^*) = 0 \quad \text{for all } x \in E\; \},$$

and the *annihilator* ${}^0F$ of F in Z, by

$${}^0F = \{x \in Z \; : \; (x, x^*) = 0 \quad \text{for all } x^* \in F\}.$$

If Z is a reflexive space and, in particular, if Z is a Hilbert space, we shall write $E^\perp$ in place of E^0; and because of the possibility of the identification of $Z^{**} := (Z^*)^*$ with Z in this case we will use the notation $F^\perp$ instead of ${}^0F$, and $E^{\perp\perp}$ for ${}^0(E^0)$. Given any subset E of a normed space Z, we have $\overline{span\, E} = {}^0(E^0)$[2]: that $\overline{span\, E} \subset {}^0(E^0)$ is obvious, while the other inclusion is a consequence of Haha-Banach's Theorem. Similarly, if $F \subset Z^*$, it is clear that $\overline{span\, F} \subset ({}^0F)^0$, but this time the inclusion may be proper if Z is not reflexive. These results imply that a vector subspace M of Z is dense in Z if, and only if, $M^0 = \{0\}$; and a vector subspace N of Z^* is dense in Z^* if, and only if, ${}^0N = 0$.

We have a very simple relation of the gap between two vector subspaces and the gap between their annihilators as shown by the following theorem.

Theorem 1.1. *For any pair M and N of vector subspaces of a normed space Z, we have*

$$\delta(M, N) = \delta(N^0, M^0), \qquad \hat{\delta}(M, N) = \hat{\delta}(M^0, N^0). \tag{1.11}$$

Proof. It makes use of the Hahn-Banach theorem and can be found in [36], page 201, Lemma 2.8. and Theorem 2.9.. □

(1.8) and (1.9) suggest that $\hat{\delta}(M, N)$ might be used to define a pseudodistance (a distance, if we limit ourselves to the set of closed subspaces of Z) between two arbitrary linear subspaces M and N of Z. But this is not possible in general since the function $\hat{\delta}$ does not satisfy the triangle inequality required of a distance function. Nevertheless, this inconvenience can be easily overcome by a slight modification of the definition of gap, thus, obtaining a pseudodistance function which is "equivalent" with $\hat{\delta}$. Hence, for all topological purposes $\hat{\delta}$ can be treated as if it were a pseudodistance (distance) in the class of all (closed) linear subspaces of Z. For more clarifications and details we invite the reader to see [36], pages 197 - 199.

However, in the case of Hilbert spaces this modification is not even necessary for if Z is a Hilbert space, $\hat{\delta}$ itself satisfies the triangle inequality and can be seen as a pseudodistance (distance) in the set of all (closed) linear subspaces of Z; this fact is a corollary of the following Theorem 1.2.

Let M be a closed subspace of a Hilbert space Z. We shall denote by P_M the orthogonal projector on M

Lemma 1.1. *Let M, N be closed subspaces of a Hilbert space Z. Then*

$$\delta(M, N) = \|(1 - P_N)P_M\|. \tag{1.12}$$

[2] *span* E is the vector subspace of Z spanned by E, that is, the smallest vector subspace of Z containing E, or equivalently, the subspace of Z of all linear combinations of the vectors of E.

Proof. Since for any $u \in H$ $\operatorname{dist}(u, N) = \|u - P_N u\|$, we have

$$\delta(M, N) = \sup_{u \in M\ \|u\| \le 1} \|u - P_N u\| = \sup_{u \in M\ \|u\| \le 1} \|(1 - P_N) P_M u\|$$

$$= \sup_{u \in H\ \|u\| = 1} \|(1 - P_N) P_M u\| = \|(1 - P_N) P_M\|. \qquad \square$$

Theorem 1.2. *If M and N are closed vector subspaces of a Hilbert space Z, then*

$$\hat{\delta}(M, N) = \|P_M - P_N\|. \tag{1.13}$$

Proof. Let us denote for brevity

$$\delta = \hat{\delta}(M, N) = \max\{\|(1 - P_N) P_M\|,\ \|(1 - P_M) P_N\|\}.$$

The second equality above is true as a consequence of Lemma 1.1. Let $u \in Z$. Then, keeping in mind that ${P_M}^2 = P_M$ and $(1 - P_M)^2 = 1 - P_M$, and using the Pythagorean Theorem,

$$\begin{aligned}
\|(P_M - P_N)u\|^2 &= \|(1 - P_N) P_M u - P_{\mathrm{N}}(1 - P_M) u\|^2 \\
&= \|(1 - P_N) P_M u\|^2 + \|P_{\mathrm{N}}(1 - P_M) u\|^2 \\
&\le \|(1 - P_N) P_M\|^2 \|P_M u\|^2 + \|P_{\mathrm{N}}(1 - P_M)\|^2 \|(1 - P_M) u\|^2 \\
&\le \delta^2 \|P_M u\|^2 + \delta^2 \|(1 - P_M) u\|^2 \\
&= \delta^2 \|u\|^2.
\end{aligned}$$

Observe that, in obtaining the inequality in the second last line, we have made use of the fact that $\|P_{\mathrm{N}}(1 - P_M)\| = \|\left(P_{\mathrm{N}}(1 - P_M)\right)^*\| = \|(1 - P_M) P_N\| \le \delta$.

In particular,

$$\max\{\|(1 - P_N) P_M u\|,\ \|P_{\mathrm{N}}(1 - P_M) u\|\} \le \|(P_M - P_N) u\| \le \delta \|u\|$$

for every $u \in Z$, which, when taking the supremum over all of $u \in Z$, with $\|u\| \le 1$, yield

$$\max\{\|(1 - P_N) P_M\|,\ \|P_{\mathrm{N}}(1 - P_M)\|\} \le \|P_M - P_N\| \le \delta.$$

But the first member in this chain of inequalities is δ, again as a consequence of the already observed fact that $\|P_{\mathrm{N}}(1 - P_M)\| = \|(1 - P_M) P_N\|$, Thus we have

$$\|P_M - P_N\| = \delta,$$

which is exactly what we wanted to prove. $\square$

We need the following lemmas to study the connection of the gap between subspaces and their dimensions - at least in the case of Hilbert spaces (see Theorem 1.3).

Lemma 1.2. *Let M and N be subspaces of a Hilbert space Z and $L = M \cap N$. Moreover, let $L_M^\perp$, $L_N^\perp$ be the orthogonal complements of L in M, N respectively. Then*

$$\delta(M, N) = \delta(L_M^\perp, L_N^\perp)\,.$$

Proof. Note that

$$\delta(M,N) = \sup_{u\in M,\,\|u\|\le 1} \|u - P_N u\|$$

$$= \sup_{u\in M,\,\|u\|\le 1} \|u - P_L u - P_{L_N^\perp} u\| = \sup_{u\in M,\,\|u\|\le 1} \left\|u - P_L u - P_{L_N^\perp}(u - P_L u)\right\|.$$

Since

$$\{u - P_L u : u \in M,\ \|u\| \le 1\} = \{v \in L_M^\perp : \|v\| \le 1\},$$

it follows that

$$\delta(M,N) = \sup_{v\in L_M^\perp,\,\|v\|\le 1} \|v - P_{L_N^\perp} v\| = \delta(L_M^\perp, L_N^\perp). \qquad \square$$

Remark 1.3. *Let in Examples 1.1, 2) $L = M\cap N$ and S be the plane passing through the origin and orthogonal to L. Then the angle between the planes M and N is equal to the angle between straight lines $M\cap S$ and $N\cap S$. By Examples 1) and 2) $\delta(M,N) = \delta(M\cap S, N\cap S)$. This also follows by the preceeding Lemma 1.2, because $M\cap S = L_M^\perp$ and $N\cap S = L_N^\perp$.*

Lemma 1.3. *Let M and N be closed linear subspaces of a Hilbert space H. Then $\dim M < \dim N$ implies*

$$\delta(N,M) = 1.$$

Proof. Without loss of generality we may assume that $\dim N < \infty$ and, by Lemma 1.2, that $M\cap N = \varnothing$.

We claim that there exists a vector $v \in N$ such that $v \perp M$ and $\|v\| = 1$. Indeed, let $\dim M = m$, $\dim N = n$, $e_1,\dots,e_m$ be a basis for M and $f_1,\dots,f_n$ be a basis for N. We are looking for a vector

$$v = \sum_{j=1}^{n} \alpha_j f_j,$$

where not all of α_j are equal to 0. The condition of v being orthogonal to M is equivalent to

$$\sum_{j=1}^{n} \alpha_j (f_j, e_i) = 0$$

for every $i = 1,\dots,n$. Since $m < n$, this system of linear equations in the unknowns $\alpha_1,\dots,\alpha_n$ has a non trivial solution. In this way we obtain a non zero vector v which is orthogonal to M and, if need be, we can normalize it so that $\|v\| = 1$. Thus we have

$$1 \ge \delta(N,M) \ge \operatorname{dist}(v,M) = \|v\| = 1. \qquad \square$$

Theorem 1.3. *Let M,N be vector subspaces of a normed space Z. $\delta(M,N) < 1$ implies $\dim M \le \dim N$ and $\hat\delta(M,N) < 1$ implies $\dim M = \dim N$.*

Proof. If Z is a Hilbert space, the result is just a consequence of the preceeding Lemma 1.3. In the general case the proof can be found in [36, Ch.IV, §2.2]. $\square$

1.2 Gap between operators

Let us now give the definition of the gap between two arbitrary linear operators. We proceed in the same spirit as in Tosio Kato's book [36], but extend the definition to general operators while Kato in his monograph [36] limits himself to the case of closed operators. Moreover, some of the results and proofs are slightly improved.

Since we are going to deal with graphs of linear operators, given two normed spaces X, Y, it will be necessary to consider the Cartesian product normed space $X \times Y$ with the usual addition, multiplication by scalars and norm defined by $\|(u,v)\|_{X\times Y} = (\|u\|_X^2 + \|v\|_Y^2)^{\frac{1}{2}}$ for all $u \in X$, $v \in Y$. If the norms of X and Y derive, respectively, from certain inner products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$, then $X \times Y$ can also be equipped with an inner product defined by $((u,v),(u',v'))_{X\times Y} = (u,u')_X + (v,v')_Y$ for all $u, u' \in X$, $v, v' \in Y$, from which the norm $\|\cdot\|_{X\times Y}$ is derived. Thus, if X, Y are Banach (Hilbert) spaces, then $X \times Y$ is also a Banach (Hilbert) space. As usual, the subscripts from the norms and inner products will be dropped whenever no possibility for ambiguity arises.

Definition 1.3. *Let X and Y be two normed spaces and let $S, T \in \mathcal{L}(X,Y)$. Then we define*

$$\delta(S,T) = \delta(G(S), G(T)), \tag{1.14}$$

$$\hat{\delta}(S,T) = \hat{\delta}(G(S), G(T)). \tag{1.15}$$

More explicitly,

$$\delta(S,T) = \sup_{\substack{u \in \mathrm{D}(S) \\ \|u\|_X^2 + \|Su\|_Y^2 = 1}} \inf_{v \in \mathrm{D}(T)} \left(\|u-v\|_X^2 + \|Su - Tv\|_Y\right)^{\frac{1}{2}}. \tag{1.16}$$

As in Remark 1.2 analogous results hold. For instance, we can change the sign " $=$ " with " $\leq$ " in the right side member of formula (1.16) without changing the result.

Example 1.1. *Let $T \in \mathcal{B}(X,Y)$. It is easily seen that*

$$\delta(T,0) = \frac{\|T\|}{(1+\|T\|^2)^{\frac{1}{2}}}.$$

In virtue of (1.18) *it follows also that $\delta(0,T) = \delta(T,0) = \hat{\delta}(T,0)$. Further results with regards to bounded operator can be found in [17].*

A linear operator T is said to be *closable* if it admits a closed extension; in this case the smallest of such extensions is called the *closure* of T and is denoted by $\bar{T}$. In terms of graphs, it is characterized by the fact that $G(\bar{T}) = \overline{G(T)}$. (the closure is intended in $X \times Y$). Thus, (1.6) implies that, if $S, T \in \mathcal{L}(X,Y)$ are closable operators, then

$$\delta(S,T) = \delta(\bar{S}, \bar{T}). \tag{1.17}$$

Let X, Y be normed spaces and let $T \in \mathcal{L}(X,Y)$ be a densely defined linear operator, that is, an operator such that $\mathrm{D}(T)$ is dense in X. Then the *adjoint* (or *dual*) *operator* of T is the operator $T^* : \mathrm{D}(T^*) \subset Y^* \to X^*$, where

$$\mathrm{D}(T^*) = \big\{ v^* \in Y^* \ : \text{ there exists } u^* \in X^* \text{ such that}$$

$$(Tu, v^*)_Y = (u, u^*)_X \text{ for all } u \in \mathrm{D}(T) \ \}$$

and

$$T^* v^* = u^*.$$

Note that it is essential for T to be densely defined in order to T^* be well-defined. Furthermore, T^* is characterized by the fact that[3] $G(T^*) = G'(T)^0$, where $G'(T) = \{(Tx, x) \ : \ x \in \mathrm{D}(T)\}$ is the *inverse graph* of T. Since an annihilator is always closed, it follows that T^* is always closed even if T is not closed or closable. We have also that $R(T)^0 = \mathrm{N}(T^*)$ and ${}^0R(T^*) = \mathrm{N}(T)$, where $\mathrm{N}(T), R(T)$ are, respectively, the null space and the range of a linear operator T.

Theorem 1.4. *Let X, Y be normed spaces and let $S, T \in \mathcal{L}(X, Y)$ be densely defined linear operators. Then*

$$\delta(S, T) = \delta(T^*, S^*), \qquad \hat{\delta}(S, T) = \hat{\delta}(S^*, T^*). \tag{1.18}$$

Proof. A direct consequence of Theorem 1.1. □

Let X be a Hilbert space and $T \in \mathcal{L}(X)$ a densely defined operator. We identify the adjoint space X^* with X. Then the operator T is said to be *symmetric* if T^* is an extension of T, that is, if $(Tu, v) = (u, Tv)$ for all $u, v \in \mathrm{D}(T)$. If $T^* = T$, then T is said to be *self-adjoint.* Symmetric operators are always closable. A symmetric operator T is said to be *essentially self-adjoint* if $\bar{T}$ is self-adjoint, or, equivalently, if $T^* = \bar{T}$. The previous Theorem 1.4 implies the following corollary.

Corollary 1.1. *Let X be a Hilbert space, and let S, T be (essentially) self-adjoint operators in X. Then*

$$\delta(S, T) = \delta(T, S) = \hat{\delta}(S, T). \tag{1.19}$$

We will continue to use the letters X, Y to denote normed spaces, unless otherwise specified, in the following part of this subsection.

Theorem 1.5. *Let $S, T \in \mathcal{L}(X, Y)$, $A \in \mathcal{B}(X, Y)$. Then*

$$\delta(S + A, T + A) \leq (2 + \|A\|^2)\delta(S, T), \tag{1.20}$$

$$\hat{\delta}(S + A, T + A) \leq (2 + \|A\|^2)\hat{\delta}(S, T). \tag{1.21}$$

Proof. Since $\mathrm{D}(A) = X$, we have $\mathrm{D}(S + A) = \mathrm{D}(S)$ and $\mathrm{D}(T + A) = \mathrm{D}(T)$. Let $u \in \mathrm{D}(S)$ and

$$\|u\|_X^2 + \|(S + A)u\|_Y^2 = 1.$$

Then

$$\|Su\|_Y \leq \|(S + A)u\|_Y + \|Au\|_Y \leq \|(S + A)u\|_Y + \|A\|\|u\|_X,$$

hence, by Cauchy-Schwarz inequality,

$$\|Su\|_Y^2 \leq (\|u\|_X^2 + \|(S + A)u\|_Y^2)(1 + \|A\|^2) = 1 + \|A\|^2$$

[3] We are identifying $(X \times Y)^*$ with $X^* \times Y^*$; any $(u^*, v^*) \in X^* \times Y^*$ is mapped in the linear form $X \times Y \to \mathbb{C}$ with $(u, v) \to (u, u^*)_X + (v, v^*)_Y$ for all $u \in X$, $v \in Y$. It can be easily verified that this mapping is an isometric linear homeomorphism of $X^* \times Y^*$ with $(X \times Y)^*$.

and

$$(\|u\|_X^2 + \|Su\|_Y^2)^{\frac{1}{2}} \le (2 + \|A\|^2)^{\frac{1}{2}}.$$

Moreover, for any $v \in \mathrm{D}(T)$,

$$\begin{aligned} \|(S+A)u - (T+A)v\|_Y &= \|Su - Tv + A(u-v)\|_Y \\ &\le \|Su - Tv\|_X + \|A\|\|u-v\|_X \end{aligned}$$

hence

$$\|(S+A)u - (T+A)v\|_Y^2 \le (\|u-v\|_X^2 + \|Su - Tv\|_Y^2)(1 + \|A\|^2)$$

and

$$(\|u-v\|_X^2 + \|(S+A)u - (T+A)v\|_Y^2)^{\frac{1}{2}} \le (2 + \|A\|^2)^{\frac{1}{2}}(\|u-v\|_X^2 + \|Su - Tv\|_Y^2)^{\frac{1}{2}}.$$

Therefore

$$\begin{aligned} \delta(S+A, T+A) &= \sup_{\substack{u \in \mathrm{D}(S) \\ \|u\|_X^2 + \|(S+A)u\|_Y^2 = 1}} \inf_{v \in \mathrm{D}(T)} \left(\|u-v\|_X^2 + \|(S+A)u - (T+A)v\|_Y^2\right)^{\frac{1}{2}} \\ &\le (2 + \|A\|^2)^{\frac{1}{2}} \sup_{\substack{u \in \mathrm{D}(S) \\ (\|u\|_X^2 + \|Su\|_Y^2)^{\frac{1}{2}} \le \\ (2+\|A\|^2)^{\frac{1}{2}}}} \inf_{v \in \mathrm{D}(T)} \left(\|u-v\|_X^2 + \|Su - Tv\|_Y^2\right)^{\frac{1}{2}} \\ &= (2 + \|A\|^2)\delta(S,T). \end{aligned}$$

Thus, inequality (1.20) holds and it also implies inequality (1.21) as a direct consequence. □

Theorem 1.6. *Let $S, T \in \mathcal{L}(X,Y)$. Assume that $T \in \mathcal{B}(X,Y)$ and*

$$\delta(S,T) < (1 + \|T\|^2)^{-1/2}.$$

Then S is also bounded (as an operator from its domain $\mathrm{D}(S)$ to Y). Furthermore,

$$\|S - T\|_{\mathrm{D}(S) \to Y} \le \frac{1 + \|T\|^2}{1 - (1 + \|T\|^2)^{1/2}\delta(S,T)}\delta(S,T). \tag{1.22}$$

Proof. *Step 1.* Let $u \in \mathrm{D}(S)$. Then for any $v \in \mathrm{D}(T)$

$$\begin{aligned} \|(S-T)u\|_Y &= \|Su - Tv - T(u-v)\|_Y \\ &\le \|Su - Tv\|_Y + \|T\|\|u-v\|_X. \end{aligned}$$

Hence, by the Cauchy-Schwarz inequality,

$$\|(S-T)u\|_Y \le \left(\|u-v\|_X^2 + \|Su - Tv\|_X^2\right)^{\frac{1}{2}} \left(1 + \|T\|^2\right)^{\frac{1}{2}}$$

and

$$\begin{aligned} \|(S-T)u\|_Y &\le \inf_{v \in \mathrm{D}(T)} \left(\|u-v\|_X^2 + \|Su - Tv\|_Y\right)^{\frac{1}{2}} \left(1 + \|T\|^2\right)^{\frac{1}{2}} \\ &\le \delta(S,T) \left(1 + \|T\|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Step 2. Assume that

$$\|u\|_X^2 + \|Su\|_Y^2 = 1. \tag{1}$$

Then

$$\begin{aligned}
1 &\le \left(\|u\|_X^2 + \|Tu + (S-T)u\|_Y^2\right)^{\frac{1}{2}} \\
&\le \left(\|u\|_X^2 + (\|T\|\|u\|_X + (S-T)u\|_Y)^2\right)^{\frac{1}{2}} \\
&= \left((1+\|T\|^2)\|u\|_X^2 + 2\|T\|\|u\|_X\|Su - Tu\|_X + \|Su-Tu\|_Y^2\right)^{\frac{1}{2}} \\
&\le \left((1+\|T\|^2)\|u\|_X^2 + 2(1+\|T\|^2)^{\frac{1}{2}}\|u\|_X\|Su - Tu\|_X + \|Su-Tu\|_Y^2\right)^{\frac{1}{2}} \\
&= (1+\|T\|^2)^{\frac{1}{2}}\|u\|_X + \|Su-Tu\|_Y.
\end{aligned}$$

Therefore

$$\|Su - Tu\|_Y \le (1+\|T\|^2)\delta(S,T)\|u\|_X + (1+\|T\|^2)^{\frac{1}{2}}\delta(S,T)\|Su-Tu\|_Y.$$

Hence for all $u \in \mathrm{D}(S)$ satisfying (1)

$$\|Su - Tu\|_Y \le \frac{1+\|T\|^2}{1-(1+\|T\|^2)^{1/2}\delta(S,T)}\delta(S,T)\|u\|_X.$$

Since this inequality is homogeneous in u, it is true for any $u \in \mathrm{D}(S)$ without normalization (1). Thus $S - T \in \widetilde{\mathcal{B}}(X,Y)$, hence $S \in \widetilde{\mathcal{B}}(X,Y)$, and inequality (1.22) holds. $\square$

Corollary 1.2. *Let X, Y be Banach spaces. If in addition to the assumptions of the previous Theorem 1.6, $S \in \mathcal{C}(X,Y)$ and $\overline{\mathrm{D}(S)} = X$, then $S \in \mathcal{B}(X,Y)$ and*

$$\|S - T\| \le \frac{1+\|T\|^2}{1-(1+\|T\|^2)^{1/2}\delta(S,T)}\delta(S,T). \tag{1.23}$$

Proof. Since by Theorem 1.6 $S \in \widetilde{\mathcal{B}}(X,Y)$ and $S \in \mathcal{C}(X,Y)$, it follows that the domain $\mathrm{D}(S)$ is closed. On the other hand $\overline{\mathrm{D}(S)} = X$, hence $S \in \mathcal{B}(X,Y)$ and estimate (1.23) follows from (1.22). $\square$

Theorem 1.7. *Let $S, T \in \mathcal{L}(X,Y)$. Assume that $T \in \mathcal{B}(X,Y)$ and*

$$\delta(T,S) < (1+\|T\|^2)^{-1/2}.$$

Then the operator S is a densely defined.

Proof. Let $v \in X$ be any vector so normalized that (v, Tv) is in unit sphere of $G(T)$;

$$\|v\|_X^2 + \|Tv\|_Y^2 = 1.$$

Let δ' be such that $\delta(T,S) < \delta' < (1+\|T\|^2)^{-1/2}$. Then there is $u \in \mathrm{D}(S)$ such that

$$\|v-u\|_X^2 + \|Tv - Su\|_Y^2 < \delta'^2.$$

Hence $\|v-u\|_X < \delta'$ and $\mathrm{dist}(v, M) < \delta'$, where M is the closure of $\mathrm{D}(S)$. Since $1 \le (1+\|T\|^2)\|v\|_X^2$, $\mathrm{dist}(v,M) < \delta'(1+\|T\|^2)^{1/2}\|v\|_X$. The last inequality is homogeneous in v and therefore true for any $v \in X$. Since $\delta'(1+\|T\|^2)^{1/2} < 1$ it follows that $M = X$; otherwise there would exist a $v \neq 0$ such that $\operatorname{dist}(v, M) > \delta'(1+\|T\|^2)^{1/2}\|v\|_X$ as a consequence of a well-known fact, which asserts that for any closed subspace $M \neq X$ and any $0 < \delta < 1$ there is a $v \in X$ such that $\|v\| = 1$ and $\operatorname{dist}(v, M) > 1 - \delta$. Thus $\mathrm{D}(S)$ is dense in X. $\square$

Let X, Y be Banach spaces. It is a known fact that if $S \in \mathcal{L}(X, Y)$ is a closed densely defined operator, then $S \in \mathcal{B}(X, Y)$ if, and only if, $S^* \in \widetilde{\mathcal{B}}(Y^*, X^*)$. For this non-trivial result we refer to Brown [8]. Taking into account this result we have the following corollary.

Corollary 1.3. *In addition to the assumptions of the preceeding Theorem 1.7, let the normed spaces X, Y be Banach spaces, and the operator S be closable, then $S \in \widetilde{\mathcal{B}}(X, Y)$, and the following estimate holds:*

$$\|S - T\|_{\mathrm{D}(S)\to Y} \leq \frac{1+\|T\|^2}{1-(1+\|T\|^2)^{1/2}\delta(T,S)}\delta(T,S). \tag{1.24}$$

Proof. To show the boundedness of S we apply Theorem 1.6 to the operator S^* which exists because S is densely defined in virtue of the previous Theorem 1.7. This can be done since $\|T^*\| = \|T\|$ and, by (1.18), $\delta(S^*, T^*) = \delta(T, S)$. Thus, we deduce that $S^* \in \widetilde{\mathcal{B}}(Y^*, X^*)$. Since $\bar{S}^* = S^*$, the premise to this corollary implies that $\bar{S} \in \mathcal{B}(X, Y)$, and therefore $S \in \widetilde{\mathcal{L}}(X, Y)$. Finally, we apply estimate (1.22) to the pair S^*, T^*, (1.18) and the fact that $\|S - T\|_{\mathrm{D}(S)\to Y} = \|S^* - T^*\|$ to obtain (1.24). $\square$

Corollary 1.4. *Let X, Y be Banach spaces. Let $S \in \mathcal{C}(X, Y)$ and $T \in \mathcal{B}(X, Y)$. Assume also that*

$$\delta(T,S) < (1+\|T\|^2)^{-1/2}.$$

Then $S \in \mathcal{B}(X, Y)$ and

$$\|S - T\| \leq \frac{1+\|T\|^2}{1-(1+\|T\|^2)^{1/2}\delta(T,S)}\delta(T,S). \tag{1.25}$$

Proof. The previous Theorem 1.7 implies that S is a densely defined and bounded operator in its domain $\mathrm{D}(S)$. On the other hand, the closed graph theorem implies that S is bounded if, and only if, $\mathrm{D}(S)$ is closed. Thus, $\mathrm{D}(S) = \overline{\mathrm{D}(S)} = X$ and therefore the operator S is defined all over X and is bounded; in our notations $S \in \mathcal{B}(X, Y)$. Then estimate (1.25) is just a direct consequence of (1.24). $\square$

These results are summarized in the following.

Theorem 1.8. *Let X, Y be Banach spaces and $S, T \in \mathcal{L}(X, Y)$. Assume that $T \in \mathcal{B}(X, Y)$ and let*

$$\delta_{\min}(S,T) = \min\{\delta(S,T),\ \delta(T,S)\} < (1+\|T\|^2)^{-1/2}.$$

Then $S \in \widetilde{\mathcal{B}}(X, Y)$ and

$$\|S - T\|_{\mathrm{D}(S)\to Y} \leq \frac{1+\|T\|^2}{1-(1+\|T\|^2)^{1/2}\delta_{\min}(S,T)}\delta_{\min}(S,T). \tag{1.26}$$

Moreover, if $\delta(T,S) < (1+\|T\|^2)^{-1/2}$, then S is densely defined.

Remark 1.4. *If in the previous theorem the operator S is assumed to be closed and densely defined, then $S \in \mathcal{B}(X, Y)$ and the norm of the difference $\|S-T\|$ is still bounded from above by the right side member of* (1.26)*.*

The gap is invariant with respect to inversion, that is, if $S, T \in \mathcal{L}(X,Y)$ are invertible, then $\delta(S^{-1}, T^{-1}) = \delta(S,T)$. This remark together with Theorem 1.6 and Theorem 1.7 imply the following important results, which, we may continue to call as *the principle of stability of bounded invertibility*, (see again [36], page 205, Theorem 2.20).

Theorem 1.9. *Let $S, T \in \mathcal{L}(X,Y)$ and assume that T is invertible with bounded inverse $T^{-1} \in \mathcal{B}(Y,X)$. We have:*

(i) If

$$\delta(S,T) < (1 + \|T^{-1}\|^2)^{-1/2},$$

then S is invertible, $S^{-1} \in \widetilde{\mathcal{B}}(Y,X)$ and

$$\|S^{-1} - T^{-1}\|_{R(S)\to X} \leq \frac{1 + \|T^{-1}\|^2}{1 - (1 + \|T^{-1}\|^2)^{1/2}\delta(S,T)}\delta(S,T). \tag{1.27}$$

(ii) If S is densely defined and

$$\hat{\delta}(T,S) < (1 + \|T^{-1}\|^2)^{-1/2},$$

then, S is invertible, $S^{-1} \in \widetilde{\mathcal{B}}(Y,X)$ and, in addition to estimate (1.27)*, the following one holds too:*

$$\|S^{-1} - T^{-1}\|_{R(S)\to X} \leq \frac{1 + \|T^{-1}\|^2}{1 - (1 + \|T^{-1}\|^2)^{1/2}\delta(T,S)}\delta(T,S). \tag{1.28}$$

Moreover, in this case the range $R(S)$ of S is dense in Y.

Proof. If we knew that S is invertible, then the assertions (i) and (ii) would be direct consequences of Theorem 1.6 and Theorem 1.7, respectively.

Thus, it remains to prove that S is invertible. This can be done directly for assertion (i); let S be such that $\delta(S,T) < (1 + \|T^{-1}\|^2)^{-1/2}$, and suppose, by contradiction, that a $u \in \mathrm{D}(S)$ such that $\|u\|_X = 1$, $Su = 0$ can be found. Then $(u, 0)$ is in the unit sphere of $G(S)$, hence there is a $v \in \mathrm{D}(T)$ such that $\|u - v\|_X^2 + \|Tv\|_Y^2 < \delta'^2$ for some number δ' with the property that $\delta(S,T) < \delta' < (1 + \|T^{-1}\|^2)^{-1/2}$. In this way we have $1 = \|u\|^2 \leq (\|u - v\|_X + \|v\|_X)^2 \leq (\|u - v\|_X + \|T^{-1}\|\|Tv\|_Y)^2 \leq (1 + \|T^{-1}\|^2)\delta'^2 < 1$, a contradiction.

Now take S as in the hypotheses of (ii). The adjoint operator S^* of S exists, since S is by assumption densely defined. Keeping in mind that $\delta(S^*, T^*) = \delta(T,S)$ and $(T^*)^{-1} = (T^{-1})^*$, we can apply (i) to the pair S^*, T^*, and thus, conclude that S^* is invertible and $(S^*)^{-1} \in \widetilde{\mathcal{B}}(X^*, Y^*)$. Since the gap is invariant under inversion, we have due to the assumptions, $\delta((S^*)^{-1}, (T^*)^{-1}) = \delta(S^*, T^*) < (1 + \|T\|^2)^{-1/2}$. Therefore, $(S^*)^{-1}$ is densely defined, and since it is also closed, we deduce that $(S^*)^{-1} \in \widetilde{\mathcal{B}}(X^*, Y^*)$. This, in particular, implies that $\mathrm{N}(S) = {}^0R(S^*) = {}^0(X^*) = 0$, so that S is invertible. $\square$

Remark 1.5. *In order to the second assertion in the previous Theorem 1.9, and in particular the estimate* (1.28)*, hold, it would suffice that* $\delta(T,S) < (1+\|T^{-1}\|^2)^{-1/2}$*, if we knew in advance that* S *is invertible. The other assumption* $\delta(S,T) < (1+\|T^{-1}\|^2)^{-1/2}$ *serves precisely for this, to guarantee that* S *is invertible like in part (i) of the foresaid theorem.*

If in part (ii) of Theorem 1.9 the spaces X, Y *are assumed to be Banach spaces and the operator* S *is assumed to be closed, then, since* S^{-1} *is closed too and densely defined, that is,* $\overline{R(S)} = Y$*, it follows, by the closed graph theorem, that* $R(S)$ *is closed and therefore* S^{-1} *is everywhere defined in* Y *and bounded (with our notations* $S^{-1} \in \mathcal{B}(Y,X)$*) and the norm of the difference* $\|S^{-1} - T^{-1}\|$ *is still majorized by the right side member of* (1.28)*.*

1.3 Estimates of the variation of eigenfunctions via the gap between operators

Now, we recall briefly those results of spectral theory of linear operators that will be useful to us. Let X be a Banach space. The symbol $\mathcal{B}(X)$ will denote the Banach algebra of the bounded linear operators in X. In what follows T is assumed to be a closed operator in X ($T \in \mathcal{C}(X)$). Then the same is true for $T - \xi$ for any complex number ξ. The set of the complex numbers ξ such that $T - \xi$ is invertible and

$$R(\xi) = R(\xi, T) = (T - \xi)^{-1} \in \mathcal{B}(X) \tag{1.29}$$

is called the *resolvent set* of T and is denoted by $\rho(T)$. The operator-valued function $R(\xi)$ thus defined on the resolvent set $\rho(T)$ is called the *resolvent*[4] and satisfies the so-called *resolvent equation*; for any $\xi_1, \xi_2 \in \rho(T)$

$$R(\xi_1) - R(\xi_2) = (\xi_1 - \xi_2) R(\xi_1) R(\xi_2). \tag{1.30}$$

Using this, it can be shown that $\rho(T)$ is an open set in the complex plane and the so-called Neumann series for the resolvent is valid; if $\xi_0 \in \rho(T)$ then

$$R(\xi) = \sum_{n=0}^{\infty} (\xi - \xi_0)^n R(\xi_0)^{n+1} \tag{1.31}$$

for any $\xi \in \mathbb{C}$ such that series on the right side is convergent. Hence $R(\xi)$ is an holomorphic function in $\rho(T)$ and it can not be continued analytically beyond the boundary of $\rho(T)$.

The complementary set $\sigma(T)$ of $\rho(T)$ (in the complex plane $\mathbb{C}$) is called the spectrum of T. In the case of X being finite dimensional $\sigma(T)$ consists of a finite set of points (eigenvalues of T), but in the general case the situation is much more complicated. It is possible for $\sigma(T)$ to be empty or to cover the whole plane $\mathbb{C}$.

If $T \in \mathcal{B}(X)$, then $\sigma(T)$ is never empty and is bounded. Actually

$$\max_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|T^n\|^{\frac{1}{n}}, \tag{1.32}$$

[4]The resolvent is the operator valued function $\xi \in \rho(T) \to R(\xi) \in \mathcal{B}(X)$. It appears, however, that also the value $R(\xi)$ of this function at a particular ξ is customarily called the resolvent. Sometimes $(\xi - T)^{-1}$ instead of $(T - \xi)^{-1}$ is called the resolvent of T.

and this number is called the spectral radius of T and is denoted by $spr(T)$.

The radius of convergence of the Neumaann series in (1.31) is $1/spr(R(\xi_0))$. Furthermore, if X is an Hilbert space and T is a normal operator (in particular, if T is self-adjoint), then

$$spr(T) = \|T\|. \tag{1.33}$$

Let us turn at the general case assuming that $T \in \mathcal{C}(X)$ only. As to the behavior of the resolvent of T at the point at infinity ∞, we have the following: if the resolvent set $\rho(T)$ contains the exterior of a disc (or equivalently, $\sigma(T)$ is bounded), then either $T \in \mathcal{B}(X)$, $R(\xi)$ is holomorphic at $\xi = \infty$ and $R(\infty) = 0$, or $R(\xi)$ has an essential singularity at ∞. In view of this fact, it is natural to include $\xi = \infty$ in the resolvent set of T if $T \in \mathcal{B}(X)$ and in the spectrum of T otherwise. Thus, we can speak about the *extended resolvent set* of T and the *extended spectrum* of T and use the notations $\widetilde{\rho}(T)$, $\widetilde{\sigma}(T)$. Hence, in this context, T is bounded ($T \in \mathcal{B}(X)$) if and only if its extended spectrum $\widetilde{\sigma}(T)$ is a bounded set of $\mathbb{C}$.

If T is an invertible operator, $\widetilde{\sigma}(T)$ and $\widetilde{\sigma}(T^{-1})$ are mapped onto each other by the mapping $\xi \to \xi^{-1}$ of the extended complex plane[5].

A consequence of this is the fact that if $\xi_0 \in \rho(T)$, then the spectrum of $R(\xi_0)$ is the bounded set obtained from $\widetilde{\sigma}(T)$ by the transformation $\xi \to \xi' = (\xi - \xi_0)^{-1}$ and using (1.32) we have

$$spr(R(\xi_0)) = \frac{1}{\mathrm{dist}(\xi_0, \sigma(T))}. \tag{1.34}$$

If the operator T is self-adjoint (or more generally, normal) $R(\xi_0)$ is normal and

$$\|R(\xi_0)\| = \frac{1}{\mathrm{dist}(\xi_0, \sigma(T))}. \tag{1.35}$$

Let λ_0 be an isolated point of $\sigma(T)$, that is an isolated singularity of $R(\xi)$. We consider the Laurent series of $R(\xi)$ at $\xi = \lambda_0$:

$$R(\xi) = \sum_{n=-\infty}^{\infty} (\xi - \lambda_0)^n A_n, \tag{1.36}$$

where

$$A_n = \frac{1}{2\pi i} \int_\Gamma \frac{R(\xi)}{(\xi - \lambda_0)^{n+1}} d\xi, \tag{1.37}$$

with Γ[6] being a positively oriented circle that enclosed λ_0 but no other point of $\sigma(T)$. Since Γ may be replaced with a slightly larger circle Γ' without changing (1.37), we have

$$\begin{aligned} A_n A_m &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_\Gamma \frac{R(\xi)R(\xi')}{(\xi - \lambda_0)^{n+1}(\xi' - \lambda_0)^{m+1}} d\xi d\xi' \\ &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma'} \int_\Gamma \frac{R(\xi) - R(\xi')}{(\xi - \lambda_0)^{n+1}(\xi' - \lambda_0)^{m+1}(\xi - \xi')} d\xi d\xi', \end{aligned}$$

[5]This is only a special case of the *spectral mapping theorem*, which asserts that given an operator T and a function φ, under certain hypothesis on T and φ, the operator $\varphi(T)$ can be defined and $\widetilde{\sigma}(\varphi(T)) = \varphi(\widetilde{\sigma}(T))$.

[6]Actually, Γ could be any simple closed rectifiable curve enclosing only λ_0 and leaving outside the rest of the spectrum of T.

where the resolvent equation (1.30) has been used. The double integral on the right may be computed in any order. Considering that Γ' lies outside Γ, we have[7]

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{1}{(\xi-\lambda_0)^{n+1}(\xi'-\xi)}d\xi=\eta_n\frac{1}{(\xi'-\lambda_0)^{n+1}},$$

$$\frac{1}{2\pi i}\int_{\Gamma'}\frac{1}{(\xi'-\lambda_0)^{m+1}(\xi'-\xi)}d\xi'=(1-\eta_n)\frac{1}{(\xi-\lambda_0)^{m+1}},$$

where the symbol η_n is defined by

$$\eta_n=1 \quad \text{for} \quad n\geq 0 \quad \text{and} \quad \eta_n=0 \quad \text{for} \quad n<0.$$

Thus,

$$A_nA_m=\frac{\eta_n+\eta_m-1}{2\pi i}\int_{C_r^+(\lambda_0)}\frac{R(\xi)}{(\xi-\lambda_0)^{n+m+2}}d\xi=(\eta_n+\eta_m-1)A_{n+m-1}. \quad (1.38)$$

For $m=n=-1$, $A_{-1}^2=-A_{-1}$. Thus, $-A_{-1}$ is a projector which we shall denote by P (or P_{λ_0} if we want to show explicitly its dependence on λ_0). For $n.m<0$, (1.38) gives $A_{-2}^2=-A_{-3}$, $A_{-2}A_{-3}=-A_{-2}^3=-A_{-4}$, ... On setting $D=D_{\lambda_0}=-A_2$, we obtain $A_{-k}=-D^{k-1}$ for $k\geq 2$. Similarly, we obtain $A_n=S^{n+1}$ for $n\geq 0$ with $S=S_{\lambda_0}=A_0$.

Thus, the Laurent series of $R(\xi)$ at $\xi=\lambda_0$ simplifies as

$$R(\xi) = -\frac{1}{\xi-\lambda_0}P-\sum_{n=1}^{\infty}\frac{1}{(\xi-\lambda_0)^{n+1}}D^n \quad (1.39)$$

$$+ \sum_{n=0}^{\infty}(\xi-\lambda_0)^nS^{n+1}. \quad (1.40)$$

Using again (1.38) with $n=-1$, $m=-2$ and then $n=-1$, $m=0$, we see that

$$PD=DP=D \quad PS=SP=0. \quad (1.41)$$

Thus $(1,56)$ and $(1,57)$ represent a decomposition of the operator $R(\xi)$ according to the decomposition $X=M_{\lambda_0}\oplus M'_{\lambda_0}=M\oplus M'$, where $M=M_{\lambda_0}=P_{\lambda_0}X$ and $M'=M'_{\lambda_0}=(1-P_{\lambda_0})X$. As the principal part of the Laurent series at an isolated singularity, the series in (1.56) is convergent for all $\xi\neq\lambda_0$, hence the spectral radius of D is zero. Such an operator is called quasinilpotent. If $M=M_{\lambda_0}$ is finite dimensional, then D is nilpotent, λ_0 is an eigenvalue of T and

$$m=m_{\lambda_0}=dimM_{\lambda_0} \quad (1.42)$$

is called the algebraic multiplicity of the eigenvalue λ_0. In this situation the principal part of the Laurent series of $R(\xi)$ is a finite sum with m_{λ_0} terms and $\xi=\lambda_0$ is a pole of order m_{λ_0} for $R(\xi)$.

In the general case, of course, the principal part of the Laurent series of $R(\xi)$ at $\xi=\lambda_0$ may be an infinite series. $P=P_{\lambda_0}$ and $D=D_{\lambda_0}$ are called the

[7] These integrals can be seen as the coefficients of the Laurent series $(\xi'-\xi)^{-1}=\sum_{n=0}^{\infty}\frac{1}{(\xi'-\lambda_0)^{n+1}}(\xi-\lambda_0)^n$, where $|\xi-\lambda_0|<|\xi'-\lambda_0|$, seen as a function of the variable ξ for the integrals of the first line, and as a function of the variable ξ' for the integrals of the second line.

eigenprojector and *eigenquasinilpotent* (if $D = D_{\lambda_0}$ is nilpotent, it is called simply *eigennilpotent*) associated to the isolated singularity λ_0. Multiplying (1.37) by T from the left and using $R(\xi)T \subset TR(\xi)$[8] $= 1+\xi R(\xi)$, we obtain $TA_n = \delta_{n0}+A_{n-1}$. This gives for $n = 0$ and $n = -1$

$$P_{\lambda_0}(T-\lambda_0) \subset (T-\lambda_0)P_{\lambda_0} = D_{\lambda_0}, \tag{1.43}$$

$$S_{\lambda_0}(T-\lambda_0) \subset (T-\lambda_0)S_{\lambda_0} = 1 - P_{\lambda_0}. \tag{1.44}$$

The holomorphic part of the Laurent expansion of $R(\xi)$ at $\xi = \lambda_0$ is called the *reduced resolvent* of T at λ_0; we denote it by $S_{\lambda_0}(\xi)$:

$$S_{\lambda_0}(\xi) = \sum_{n=0}^{\infty} (\xi - \lambda_0)^n S^{n+1}. \tag{1.45}$$

From this, (1.41), (1.43) and (1.44), it follows

$$S_{\lambda_0}(\lambda_0) = S_{\lambda_0}, \quad S_{\lambda_0}(\xi)P_{\lambda_0} = P_{\lambda_0}S_{\lambda_0}(\xi) = 0, \tag{1.46}$$

$$S_{\lambda_0}(\xi)(T-\xi) \subset (T-\xi)S_{\lambda_0}(\xi) = 1 - P_{\lambda_0}. \tag{1.47}$$

The last equalities show that the parts of $T - \xi$ and of $S_{\lambda_0}(\xi)$ in the invariant subspace $M' = M'_{\lambda_0}$ are the inverse of each other.

These results can be extended to the case in which we consider a finite number of isolated points $\lambda_1, \ldots, \lambda_s$ of $\sigma(T)$. We obtain

$$R(\xi) = -\sum_{k=1}^{s} \left[\frac{P_k}{\xi - \lambda_k} + \sum_{n=1}^{\infty} \frac{D_k^n}{(\xi - \lambda_k)^{n+1}} \right] + R_0(\xi). \tag{1.48}$$

Here P_k and D_k are the eigenprojectors and eigenquasinilpotents associated to the isolated points λ_k of $\sigma(T)$; they satisfy the relations

$$P_h P_k = \delta_{hk} P_h \quad P_k D_k = D_k P_k = D_k \quad (T - \lambda_k) P_k = D_k. \tag{1.49}$$

$R_0(\xi)$ is holomorphic at $\xi = \lambda_k$, $k = 1, \ldots, s$, and

$$R_0(\xi) = P_0 R(\xi) = R(\xi) P_0 \quad P_0 = 1 - (P_1 + \cdots + P_s).$$

Again, λ_k is an eigenvalue if $M_k = P_k X$ is finite dimensional with the dimension of this subspace being its algebraic multiplicity.. If $\lambda_1, \ldots, \lambda_k$ are eigenvalues of finite multiplicity, they will be called a *finite system of eigenvalues of* T. For finite system of eigenvalues, the situation is much the same as in the case of finite dimensional X. For instance, they depend continuously on T. We have further

$$TP = \sum_{k=1}^{s} (\lambda_k P_k + D_k), \quad P = P_1 + \cdots P_s. \tag{1.50}$$

Here we have what might be called a *a spectral representation* of T in a restricted sense. This is not as complete as in the case of finite dimensional X, since it does not regard T but TP.

[8]This means that $TR(\xi)$ is an extension of $R(\xi)T$

Given a closed operator T and $\lambda = \lambda[T]$ an isolated point of $\sigma(T)$, we would like to study the dependence on T of the eigenprojector $P[T] = P_\lambda[T]$ associated to $\lambda[T]$. If Γ is a closed curve around λ_0 that encloses no other point of $\sigma(T)$ and if S is a slight perturbation of T ($\hat{\delta}(S,T)$ is suitably small), then $\sigma(S)$ might have more then one point–it has at least one–or even infinitely many points, near λ_0 enclosed by the curve Γ. As an example, if $\lambda[T]$ is an eigenvalue of T of finite algebraic multiplicity m, the state of things is alike to the case of matrices: it usually splits in $1 \leq k = k[S] \leq m$ distinct eigenvalues $\lambda_1[S], \cdots, \lambda_0 k[S]$ whose sum of the algebraic multiplicities equals m and all of them are enclosed by Γ. These considerations lead us to deal with groups of spectral values, isolated from the rest of the spectrum, simultaneously. Thus, we have to consider a projector $P[S]$ that regards all that part of $\sigma(S)$ enclosed by the curve Γ. More precisely, we have the following theorem.

Theorem 1.10. *Let $T \in \mathcal{C}(X)$. Assume that $\sigma(T)$ is separated into two parts σ_0 and σ' by a rectifiable simple closed curve (or, more generally, a finite number of such curves with no point of any of these curves contained in the interior of any other) in such a way that it encloses an open set containing σ_0 in its interior and σ' in its exterior. Let*

$$P = P[T] = -\frac{1}{2\pi i}\int_\Gamma R(\xi)d\xi. \tag{1.51}$$

Then the following facts hold:

(i) P is independent from the curve (or sequence of curves) Γ used to define it, provided that Γ satisfies the above mentioned requisites. More precisely, if Γ' is another rectifiable simple curve (or finite sequence of such curves with no point of any of these curves enclosed by any other curve) that encloses an open set containing σ_0 in its interior and σ' in its exterior, then the second member of (1.51), with Γ' instead of Γ, yields the same operator P.

(ii) $P \in \mathcal{B}(X)$ and $P^2 = P$. Thus P is a (not necessarily orthogonal) projector.

(iii) If $M = M[T] = P[T]X$ and $N = N[T] = (1 - P[T])X$, then M, N are complementary (not necessarily orthogonal) closed subspaces[9]- briefly we write $X = M \dotplus N$. Moreover, M, N are T-invariant subspaces of X in the following precise sense: $N \subset \mathrm{D}(T)$, $TM \subset M$, $\mathrm{D}(T) \cap N$ is dense in N and $T(\mathrm{D}(T) \cap N) \subset N$.

(iv) The operators that T induces in M and in N that we denote with T_M and T_N are such that $\sigma(T_M) = \sigma_0$ and $\sigma(T_N) = \sigma'$. One has also that $T_{M_0} \in \mathcal{B}(M_0)$. (Thus, $\widetilde{\sigma}(T_{M_0}) = \sigma_0$ whereas $\widetilde{\sigma}(T_{M'})$ may contain ∞.)

If in addition the operator T is normal, and in particular, self-adjoint, then the following facts hold too:

(v) P is an orthogonal projector and M, N are orthogonal complements to each other; briefly this fact is written as $X = M \oplus N$.

[9] We recall that this means that $X = M + N$ and $M \cap N = \{0\}$.

(vi) *If $\sigma_0 = \{\lambda_0\}$, that is λ_0 is an isolated point of the spectrum of T, then λ_0 is an eigenvalue of T and $M[T]$ is the relative eigenspace, that is, $M[T] = \mathrm{N}(T - \lambda_0)$. In this case $\dim M[T]$ is called the* multiplicity[10] *of T.*

(vii) *If $m = \dim M[T] < \infty$, then σ_0 has at most m distinct points which are all isolated eigenvalues of T and whose sum of the respective multiplicites is exactly m. If say $\sigma_0 = \{\lambda_1, \ldots, \lambda_p\}$ ($1 \leq p \leq m$), then*

$$M[T] = \bigoplus_{i=1}^{p} \mathrm{N}(T - \lambda_i). \tag{1.52}$$

The proof of these facts is not difficult and part of it can be found in [36], Ch. III, §6.4-§6.5 pages 178-181; Ch.V, §3.5, pages 272-274; and in [53], Theorem XII.5, Theorem XII.6, Vol. IV, pages 11-13.

Another relation that might be useful in applications is the following:

$$P[T]T \subset TP[T] = -\frac{1}{2\pi i}\int_\Gamma \xi R(\xi) d\xi \in \mathcal{B}(X). \tag{1.53}$$

Given two operators S, T, it is our intention to give an estimate of

$$\hat{\delta}(M[S], M[T]) = \|P[S] - P[T]\|,$$

(see Theorem 1.2), when $\tilde{\delta}(S,T)$ is sufficiently small.

Theorem 1.11. *Let X be a Hilbert space and T a self-adjoint operator in X. Let λ_0 be an eigenvalue of T and assume that it is an isolated point of $\sigma(T)$. Precisely, assume that it is the center of a circle of radius $r > 0$ that we denote by $C_r^+(\lambda_0)$ and that does not enclose any other point of $\sigma(T)$ except λ_0. Assume also that any other point of $\sigma(T)$ different from λ_0 has distance from λ_0 not less than*[11] *$2r$. Set*

$$\delta = \frac{r}{2[2 + (|\lambda_0| + r)^2]\,(1 + r^2)^{1/2}}.$$

Any $S \in \mathcal{C}(X)$ with $\tilde{\delta}(S,T) \leq \delta$ has spectrum $\sigma(S)$ separated by $C_r^+(\lambda_0)$ into two parts $\sigma_0(S)$ and $\sigma'(S)$ ($C_r^+(\lambda_0)$ running in $P(S)$) and

$$\hat{\delta}(M[S], M[T]) \;\leq\; 2[2 + (|\lambda_0| + r)^2]\,(1 + r^2)^{1/2}\,\delta_{min}(S,T). \tag{1.54}$$

Proof. A point of the circle belongs to the resolvent set of S if $S - \xi$ is invertible with the inverse in $\mathcal{B}(X)$, and this, in particular, in virtu of Theorem 1.9, ii), Remark 1.5 and Theorem 1.5, is true if $\hat{\delta}(S - \xi, T - \xi) \leq (2 + |\xi|^2)\hat{\delta}(S,T) <$

[10] In the case of normal operators, the usual notions of algebraic and geometric multiplicities, which are found in the spectral theory literature, coincide. This is why we speak simply about *the multiplicity*.

[11] This hypothesis guaranties that $\mathrm{dist}(\xi, \sigma(T)) = |\xi - \lambda_0| = r$ for any point ξ of the circle $C_r^+(\lambda_0)$. Hence, (1.35) implies $\|R(\xi,T)\| = 1/r$ for such points $\xi \in C_r^+(\lambda_0)$.

$(1+\|R(\xi,T)\|^2)^{-1/2}$. That is, if

$$\begin{aligned}
\hat{\delta}(S,T) &< \min_{\xi\in C_r^+(\lambda_0)} \frac{1}{(2+|\xi|^2)(1+\|R(\xi,T)\|^2)^{1/2}} \\
&= \frac{1}{\max_{\xi\in C_r^+(\lambda_0)}\left[(2+|\xi|^2)(1+\|R(\xi,T)\|^2)^{1/2}\right]} \\
&= \frac{1}{\max_{\xi\in C_r^+(\lambda_0)}\left[(2+|\xi|^2)(1+\frac{1}{r^2})^{1/2}\right]} \\
&= \frac{1}{\left[2+(|\lambda_0|+r)^2\right]\left(1+\frac{1}{r^2}\right)^{1/2}} = \quad 2\delta,
\end{aligned}$$

(here we have used (1.35)). Thus, since this is the case by assumption (actually $\tilde{\delta}(S,T)\le\delta<2\delta$) we deduce that the circle $C_r^+(\lambda_0)$ lies in the resolvent set of S.

Using (1.51), (1.35) and Theorem 1.9, we obtain

$$\begin{aligned}
\|P[S]-P[T]\| &\le \frac{1}{2\pi}\int_{C_r^+(\lambda_0)} \|R(\xi,S)-R(\xi,T)\|d\xi \\
&\le r \max_{\xi\in C_r^+(\lambda_0)} \|R(\xi,S)-R(\xi,T)\| \\
&\le r \max_{\xi\in C_r^+(\lambda_0)} \frac{1+\|R(\xi,T)\|^2}{1-(1+\|R(\xi,T)\|^2)^{1/2}\delta_{\min}(S-\xi,T-\xi)}\delta_{\min}(S-\xi,T-\xi) \\
&\le r \max_{\xi\in C_r^+(\lambda_0)} \frac{(2+|\xi|^2)(1+\frac{1}{r^2})}{1-(2+|\xi|^2)(1+\frac{1}{r^2})^{1/2}\delta_{\min}(S,T)}\delta_{\min}(S,T) \\
&= \frac{r[2+(|\lambda_0|+r)^2]\left(1+\frac{1}{r^2}\right)}{1-[2+(|\lambda_0|+r)^2]\left(1+\frac{1}{r^2}\right)^{1/2}\delta_{\min}(S,T)}\delta_{\min}(S,T) \\
&= \frac{r}{2\delta-\delta_{\min}(S,T)}\delta_{\min}(S,T) \\
&\le \frac{r}{\delta}\delta_{\min}(S,T) = 2[2+(|\lambda_0|+r)^2]\left(1+r^2\right)^{1/2}\delta_{\min}(S,T)\,.
\end{aligned}$$

Then (1.54) follows from Lemma 1.2. □

Remark 1.6. *Under the assumptions of the previous Theorem 1.11, let* $|\lambda-\lambda_0|<r$, $\lambda\ne\lambda_0$. *As we saw in the proof of that theorem,* $\lambda\in\sigma(S)$ *implies*

$$(1+\|R(\lambda,T)\|^2)^{-1/2}\le(2+|\lambda|^2)\hat{\delta}(S,T).$$

In consideration of (1.35) *and the assumptions of the theorem we find that*

$$|\lambda-\lambda_0|\le\frac{(2+|\lambda|^2)\hat{\delta}(S,T)}{\sqrt{1-(2+|\lambda|^2)^2\hat{\delta}(S,T)^2}}, \tag{1.55}$$

for all $\lambda\in\sigma(S)\cap D_r(\lambda_0)$.

We are going to apply the estimates for the projectors of the previous theorem to obtain estimates for the eigenfunctions. Hence the following result, which can be found in [6] comes out to be of importance.

Lemma 1.4. *Let M and N be finite dimensional subspaces of a Hilbert space X, $\dim M = \dim N = m$, and let $u_1, \dots, u_m$ be an orthonormal basis for M. Then there exists an orthonormal basis $v_1, \dots, v_m$ for N such that*

$$\|u_k - v_k\| \leq 5^k \hat{\delta}(M, N), \qquad k = 1, \dots, m. \tag{1.56}$$

Proof. In [6], page 20, Lemma 5.4. states precisely what we want to prove, with the only difference that we have (1.56) with $\|P_M - P_N\|$, (where P_M and P_N are the orthogonal projectors of M and N respectively) instead of $\hat{\delta}(M, N)$. But these quantities are equal in virtue of Theorem 1.2, hence we have the desired result. □

Combining the above results we obtain the following theorem.

Theorem 1.12. *Let X be a Hilbert space and T, S self-adjoint (or more generally, normal) operators. Let λ_0, r, d be as in the assumptions of Theorem 1.11. Moreover, assume that the eigenvalue λ_0 of T has multiplicity m. Then, if $\hat{\delta}(S, T) < d$, there are at most m distinct eigenvalues of S in the disc $D_r(\lambda_0)$ such that the sum of the relative multiplicities is exactly m. Moreover, if $\varphi_1[S], \dots, \varphi_m[S]$ is a an orthonormal set of eigenfunctions of S corresponding to the aforementioned set of eigenvalues of S, then there exists an orthonormal set of eigenfunctions $\varphi_1[T], \dots, \varphi_m[T]$ of T corresponding to the eigenvalue λ_0 such that*

$$\|\varphi_k[S] - \varphi_k[T]\| \leq 2 \cdot 5^k [2 + (|\lambda_0| + r)^2] \left(1 + r^2\right)^{1/2} \delta_{\min}(S, T) \tag{1.57}$$

for each $k = 1, \dots, m$.

An important remark is to be done:

Remark 1.7. *The previous result is not symmetric in the operators T, S in the sense that we are obliged to fix first the eigenfunctions of S and then we can choose eigenfunctions of T (as near to those of S as we want provided that $\hat{\delta}(S, T)$ is sufficiently small), but not conversely. The fact is that the subspaces*

$$M[T] = \mathrm{N}(T - \lambda_0) \qquad \text{and} \qquad M[S] = \bigoplus_{i=1}^{p} \mathrm{N}(S - \lambda_i[S]),$$

where $\lambda_1[S], \dots, \lambda_p[S]$ are the distinct eigenvalues of S, are indeed near to each other (provided that $\hat{\delta}(S, T)$ is sufficiently small, see Theorem 1.11) but we do not have the necessary freedom in choosing the eigenfunctions of S.

2 Preliminaries about non-negative self-adjoint operators

2.1 Square roots of non-negative self-adjoint operators

Theorem 2.1. *For any non-negative self-adjoint operator S on a Hilbert space X there exists a uniquely defined non-negative self-adjoint operator $S^{1/2}$ on X, the square root of S, such that*

(i) $\mathrm{D}(S) \subset \mathrm{D}(S^{1/2}) \subset X$,

(ii) $u \in \mathrm{D}(S)$ *if and only if* $u \in \mathrm{D}(S^{1/2})$ *and* $(S^{1/2}u) \in \mathrm{D}(S^{1/2})$,

(iii) $S^{1/2}(S^{1/2}u) = Su$ *for* $u \in \mathrm{D}(S)$.

Corollary 2.1. $u \in \mathrm{D}(S)$ *if and only if* $u \in \mathrm{D}(S^{1/2})$ *and there exists* $f \in X$ *such that for all* $v \in \mathrm{D}(S^{1/2})$

$$(S^{1/2}u, S^{1/2}v) = (f, v)\,. \tag{2.1}$$

In such a case $Su = f$.

Proof. If $u \in \mathrm{D}(S)$ then, by the self-adjointness of $S^{1/2}$, (ii) and (iii), for all $v \in \mathrm{D}(S^{1/2})$.

$$(S^{1/2}u, S^{1/2}v) = (Su, v)\,. \tag{2.2}$$

Vice versa if $u \in \mathrm{D}(S^{1/2})$ and (2.1) holds, by the definition of the adjoint, it follows that $S^{1/2}u \in \mathrm{D}((S^{1/2})^*)$ and $(S^{1/2})^*(S^{1/2}u) = f$. Since $(S^{1/2})^* = S^{1/2}$, by (ii) and (iii) it follows that $u \in \mathrm{D}(S)$ and $Su = f$. □

There are different ways of describing $S^{1/2}$ but no simple ones.

(i)
$$S^{1/2} = \frac{2}{\sqrt{\pi}} \int_0^\infty S e^{-t^2 S}\, dt\,.$$

Note that, for every $x \geq 0$,

$$\sqrt{x} = \frac{2}{\sqrt{\pi}} \int_0^\infty x e^{-t^2 x}\, dt\,.$$

(ii) If $S \geq a > 0$ (*i.e.* $(Su, u) \geq a\|u\|^2$ for $u \in \mathrm{D}(S)$) then

$$S^{1/2} = \left(\frac{1}{\pi}\int_0^\infty t^{-\frac{1}{2}}(S + tI)^{-1}\, dt\right)^{-1}.$$

Note that, for $x > 0$,

$$x^{-\frac{1}{2}} = \frac{1}{\pi}\int_0^\infty t^{-\frac{1}{2}}(x + t)^{-1}\, dt\,.$$

(iii) Assume that S has compact resolvent, let λ_n, $n \in \mathbb{N}$, be the eigenvalues (arranged in non-decreasing order), ϕ_n, $n \in \mathbb{N}$ be the corresponding eigenvectors and $\{\phi_n\}_{n\in\mathbb{N}}$ be an orthonormal basis for X (see §2.3, in particular, Theorem 2.6). Then

$$\begin{aligned} S^{1/2}u &= \sum_{n=1}^{\infty} \lambda^{1/2}(u, \phi_n)\phi_n \,, \\ \mathrm{D}(S^{1/2}) &= \left\{ u \in X \,:\, \sum_{n=1}^{\infty} \lambda_n |(u, \phi_n)|^2 < \infty \right\} \end{aligned}$$

and

$$\mathrm{D}(S) = \left\{ u \in X \,:\, \sum_{n=1}^{\infty} \lambda_n^2 |(u, \phi_n)|^2 < \infty \right\} .$$

Example 2.1. *Let S be as in 3, we have*

$$\begin{aligned} S^{1/2}(S^{1/2}u) = \sum_{n=1}^{\infty} \lambda_n^{1/2}(S^{1/2}u, \phi_n)\phi_n = \sum_{n=1}^{\infty} \lambda_n^{1/2} \left(\sum_{k=1}^{\infty} \lambda_k^{1/2}(u, \phi_k)\phi_k \right) \phi_n \\ = \sum_{n=1}^{\infty} \lambda_n^{1/2} \lambda_n^{1/2} (u, \phi_n)\phi_n = Su \,. \end{aligned}$$

One verifies that $u \in \mathrm{D}(S)$ if and only if $u \in \mathrm{D}(S^{1/2})$ and $S^{1/2}u \in \mathrm{D}(S^{1/2})$.

2.2 Variationally defined non-negative self-adjoint operators, closed forms, Friedrich's extensions

As Davies has pointed out in [20], quadratic forms have proven to be extremely powerful in the study of second order elliptic equations. The fact is that it is often better to study the quadratic form associated with a non-negative self-adjoint operator S than the operator itself. Actually, it is usually easier to determine or characterize the domain of the form than that of the operator. Not only but it also happens that many differential operators with quite different domains have quadratic forms with the same domain. Hence, for the convenience of the reader, we report here the main facts that concern non-negative closable quadratic forms. The proofs are usually not provided, but we refer the reader to [20] instead.

Definition 2.1. *Let X be a Hilbert space, $\mathrm{D}(Q') \subset X$ and*

$$Q' : \mathrm{D}(Q') \times \mathrm{D}(Q') \to \mathbb{C} \,.$$

We say that Q' is a non-negative sesquilinear form *on X, if $\mathrm{D}(Q')$ is a dense linear subset of X and*

(i) $Q'(u, v)$ is linear in u,

(ii) $Q'(u, v)$ is conjugate linear in v,

(iii) $Q'(u, v) = \overline{Q'(v, u)}$, for $u, v \in \mathrm{D}(Q')$,

(iv) $Q'(u, u) \geq 0$, for $u \in \mathrm{D}(Q')$.

The quadratic form

$$Q : \mathrm{D}(Q) \to \mathbb{R}$$

defined by $\mathrm{D}(Q) = \mathrm{D}(Q')$ *and* $Q(u) = Q'(u,u)$ *for* $u \in \mathrm{D}(Q)$ *is called the* quadratic form associated *with* Q'. [12]

Recall that

$$Q'(u,v) = \frac{1}{4}\left(Q(u+v) + Q(u-v) + i(Q(u+iv) + Q(u-iv))\right).$$

Example 2.2. *Let S be a non-negative symmetric linear operator on X and define $\mathrm{D}(Q') = \mathrm{D}(S)$ and $Q'(u,v) = (Su, v)$ for $u, v \in \mathrm{D}(S)$. The Q' satisfies (i)–(iv).*

Example 2.3. *Let S be a non-negative self-adjoint operator on X.*

Define $\mathrm{D}(Q') = \mathrm{D}(S^{1/2})$ and $Q'(u,v) = (S^{1/2}u, S^{1/2}v)$, $u, v \in \mathrm{D}(S^{1/2})$. Then Q' satisfies (i)–(iv).

In the last case we say that Q' is the associated sesquilinear form which arises from S.

Definition 2.2. *The quadratic form Q on X is* closed *if $\mathrm{D}(Q)$ is complete for the norm defined by*

$$|\!|\!| u |\!|\!| = \left(\|u\|^2 + Q(u)\right)^{1/2}.$$

That is to say that, if $u_n \in \mathrm{D}(Q)$ and

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0, \qquad \lim_{m,n\to\infty} Q(u_m - u_n) = 0,$$

then there exists $u \in \mathrm{D}(Q)$ such that

$$\lim_{n\to\infty} \|u_n - u\| = 0 \qquad \text{and} \qquad \lim_{n\to\infty} Q(u_n - u) = 0.$$

Recall that $S : \mathrm{D}(S) \to X$ is closed if, $u, v \in X$, $u_n \in \mathrm{D}(S)$, $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\lim_{n\to\infty} \|Hu_n - v\| = 0$, implies $u \in \mathrm{D}(S)$ and $Su = v$. That is to say that S is closed if $\mathrm{D}(S)$ is complete for the norm

$$|\!|\!| u |\!|\!| = \left(\|u\|^2 + \|Su\|^2\right)^{1/2}.$$

Indeed, if $\mathrm{D}(S)$ is complete for $|\!|\!| \cdot |\!|\!|$ let $u, v \in X$, $u_n \in \mathrm{D}(S)$, $\lim_{n\to\infty} \|u_n - u\| = 0$ and $\lim_{n\to\infty} \|Su_n - v\| = 0$. Then

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0 \quad \text{and} \quad \lim_{m,n\to\infty} \|Su_m - Su_n\| = 0.$$

Hence there exists $\tilde{u} \in \mathrm{D}(S)$ and $\tilde{v} \in X$ such that $u_m \to \tilde{u}$ and $Su_m \to \tilde{v}$ in X. Thus $u = \tilde{u}$, $v = \tilde{v}$, $u \in \mathrm{D}(S)$ and $Su = v$.

Now let S be closed, let $u_n \in \mathrm{D}(S)$,

$$\lim_{m,n\to\infty} \|u_m - u_n\| = 0 \quad \text{and} \quad \lim_{m,n\to\infty} \|Su_m - Su_n\| = 0.$$

Then there exist $u, v \in X$ such that $u_n \to u$, $Su_n \to v$. Hence $u \in \mathrm{D}(S)$ and $Su = v$, so

$$\lim_{n\to\infty} |\!|\!| u_n - u |\!|\!| = 0.$$

The following lemma can be proved by applying Theorem 2.1.

[12] Sometimes it is assumed that $\mathrm{D}(Q) = X$, $Q(u) = Q'(u,u)$ if $u \in \mathrm{D}(Q')$ and $Q(u) = \infty$ if $u \in X \setminus Q'$.

Lemma 2.1. *If a closed quadratic form Q arises from S, then $u \in \mathrm{D}(S)$ if and only if $u \in \mathrm{D}(Q)$ and there exists $f \in X$ such that for all $v \in \mathrm{D}(Q)$*

$$Q'(u,v) = (f,v).$$

In such a case $Su = f$.

Theorem 2.2. *The following statements are equivalent.*

(i) Q is a closed quadratic form on X.

(ii) There exists a closed non-negative self-adjoint operator S on X such that Q arises from S, i.e. $\mathrm{D}(Q) = \mathrm{D}(S^{1/2})$ and $Q'(u,v) \equiv (S^{1/2}u, S^{1/2}v)$ for $u, v \in \mathrm{D}(S)$.

Definition 2.3. *A quadratic form Q on X is* closable *if it has a closed extension and the smallest closed extension is called its closure $\bar{Q}$.*

Corollary 2.2. *Let a quadratic form Q on X be closable, then there exists a closed non-negative self-adjoint operator $\overline{S}$ on X such that $\bar{Q}$ arises from $\overline{S}$.*

Theorem 2.3. *(i) A quadratic form Q on X is closable if and only if the conditions*

$$u_n \to 0 \text{ in } X \quad \text{and} \quad \lim_{m,n\to\infty} Q(u_m - u_n) = 0\,,$$

implies that $\lim_{n\to\infty} Q(u_n) = 0$.

(ii) If this condition is satisfied, then the closure $\bar{Q}$ of Q is defined in the following way.

$$\mathrm{D}(\bar{Q}) = \{u \in X \;:\; \text{there exists } u_n \in \mathrm{D}(Q) \text{ for which } u_n \to u \text{ in } X, \text{ and } Q(u_m - u_n) \to 0 \text{ as } m,n \to \infty\}$$

and $\bar{Q}'(u,v) = \lim_{n\to\infty} Q'(u_n, v_n)$ for $u, v \in \mathrm{D}(\bar{Q})$ and for all sequences u_n and v_n.

Theorem 2.4. *Let S be a non-negative symmetric operator on X. Consider the quadratic form Q defined by $\mathrm{D}(Q) = \mathrm{D}(S)$ and $Q'(u,v) = (Su,v)$, $u, v \in \mathrm{D}(S)$. Then Q is closable. Since there exists a non-negative self-adjoint operator S^F on X which is an extension of S and is such that the closure $\bar{Q}$ of Q arises from S^F, i.e. $\mathrm{D}(Q) = \mathrm{D}(S^F)^{1/2}$ and $\bar{Q}'(u,v) = \big((S^F)^{1/2}u, (S^F)^{1/2}v\big)$ for $u, v \in \mathrm{D}(S^F)^{1/2}$.*

Definition 2.4. *Let S be a non-negative symmetric operator on S, then the uniquely defined non-negative self adjoint operator S^F in the previous theorem is called the* Friedrich's extension *of S.*

2.3 Spectral properties of non-negative self-adjoint operators with compact resolvent

Definition 2.5. The point spectrum $\sigma_p(S)$ *of an operator S is the set of all eigenvalues of S. The* discrete spectrum *of S is the set of all isolated eigenvalues of S (as points of the spectrum of S) of finite multiplicity. The* essential spectrum*,* EssSpec(S)*, is the non-discrete part of the spectrum.*

The terminology, related to the essential spectrum, is "not stable". Several, non-equivalent, definitions for this notion are used in mathematical literature. (See [22, §1.4] where this is discussed in detail.)

Thus the following phrases have the same meaning:

(i) spectrum of S is discrete,

(ii) spectrum of S consists only of isolated eigenvalues of finite multiplicity,

(iii) essential spectrum of S is empty.

We recall that a symmetric linear operator S on a linear space X is *non-negative* if $(Su, u) \geq 0$ for all $u \in \mathrm{D}(S)$.

Lemma 2.2. *A self-adjoint linear operator S on a Hilbert space X is non negative if and only if $\sigma(S) \subset [0, \infty)$.*

Definition 2.6. *A operator $S \in \mathcal{L}(X)$ on a Hilbert space* has compact resolvent *if there exists $\lambda \in \mathbb{C}$ such that the operator $(S - \lambda I)^{-1}$ is compact.*

Remark 2.1. *Note that, since in this definition $\sigma(S) \neq \mathbb{C}$, $S \in \mathcal{C}(X)$. Due to the resolvent identity* (1.30)*, the above definition is equivalent to the following: A linear operator S on a Hilbert space has compact resolvent if $\sigma(S) \neq \mathbb{C}$ and the operator $(S - \lambda I)^{-1}$ is compact for all λ in the resolvent set $\mathbb{C} \setminus \sigma(S)$.*

Theorem 2.5. *Let S be a linear operator on a Hilbert space X such that $\sigma(S) \neq \mathbb{C}$. Then the following statements are equivalent.*

(i) S has compact support.

(ii) The embedding $\mathrm{D}(S) \subset X$, where $\mathrm{D}(S)$ is endowed with the graph norm $\|\|u\|\| = (\|u\|^2 + \|Su\|^2)^{1/2}$, is compact. [13]

Proof. Let $\lambda \in \mathbb{C} \setminus \sigma(S)$.

(i)⇒(ii) The operator $S - \lambda I : (\mathrm{D}(S), \|\|\cdot\|\|) \to (X, \|\cdot\|)$ is bounded since

$$\|S - \lambda I\| = \sup_{\substack{u \in \mathrm{D}(S) \\ u \neq 0}} \frac{\|Su - \lambda u\|}{(\|u\|^2 + \|Su\|^2)^{1/2}} \leq (1 + |\lambda|^2)^{1/2}$$

because

$$\begin{aligned} \|Su - \lambda u\|^2 &\leq (|\lambda|\|u\| + \|Su\|)^2 = |\lambda|^2\|u\|^2 + 2|\lambda|\|u\|\|Su\| + \|Su\|^2 \\ &\leq |\lambda|^2\|u\|^2 + \|u\|^2 + |\lambda|^2\|Su\|^2 + \|Su\|^2 = (1 + |\lambda|^2)(\|u\|^2 + \|Su\|^2)\,. \end{aligned}$$

Since $\lambda \notin \sigma(S)$, the operator $(S-\lambda I)^{-1} : X \to (\mathrm{D}(S), \|\cdot\|)$ is compact. Therefore the embedding (identity) operator $I : (\mathrm{D}(S), \|\|\cdot\|\|) \to (X, \|\cdot\|)$, which can be represented as

$$\begin{aligned} I_{(\mathrm{D}(S),\|\|\cdot\|\|)\to(X,\|\cdot\|)} &= I_{(\mathrm{D}(S),\|\|\cdot\|\|)\to(\mathrm{D}(S),\|\cdot\|)} \\ &= (S - \lambda I)^{-1}_{X\to(\mathrm{D}(S),\|\cdot\|)} (S - \lambda I)_{(\mathrm{D}(S),\|\|\cdot\|\|)\to(X,\|\cdot\|)}\,, \end{aligned}$$

[13] *I.e.* the embedding (identity) operator $I : (\mathrm{D}(S), \|\|\cdot\|\|) \to (X, \|\cdot\|)$ is compact, or equivalently for each sequence $u_k \in \mathrm{D}(S)$ such that the numerical sequence $\|u_k\|^2 + \|Su_k\|^2$ is bounded there exist $u \in X$ and a subsequence u_{k_s} such that $\lim_{s\to\infty} \|u_{k_s} - u\| = 0$. (Since I is continuous, $u \in \mathrm{D}(S)$.)

is compact, being the product of a bounded and a compact operators.

(ii)⇒(i) Since $\lambda \in \sigma(S)$, the operator $(S-\lambda I)^{1/2} : (X, \|\cdot\|) \to (\mathrm{D}(S), \|\cdot\|)$ is bounded. It is also bounded as $(S-\lambda I)^{-1} : (X, \|\cdot\|) \to (\mathrm{D}(S), \|\!|\cdot|\!\|)$. Indeed, for all $u \in X$,

$$\begin{aligned}
\|\!|(S-\lambda I)^{-1}u|\!\| &= \left(\|(S-\lambda I)^{-1}u\|^2 + \|S(S-\lambda I)^{-1}u\|^2\right)^{1/2} \\
&\le \|(S-\lambda I)^{-1}u\| + \|(S-\lambda I+\lambda I)(S-\lambda I)^{-1}u\| \\
&= \|(S-\lambda I)^{-1}u\| + \|u + \lambda(S-\lambda I)^{-1}u\| \\
&\le \|u\| + (1+|\lambda|)\|(S-\lambda I)^{-1}u\|\,.
\end{aligned}$$

Hence

$$\begin{aligned}
&\|(S-\lambda I)^{-1}u\|_{(X,\|\cdot\|)\to(\mathrm{D}(S),\|\cdot\|)} \\
&\qquad \le 1 + (1+|\lambda|)\|(S-\lambda I)^{-1}\|_{(X,\|\cdot\|)\to(\mathrm{D}(S),\|\cdot\|)} < \infty
\end{aligned}$$

Therefore, the operator $(S-\lambda I)^{-1} : (X, \|\cdot\|) \to (\mathrm{D}(S), \|\cdot\|)$ which can be represented as

$$\begin{aligned}
&(S-\lambda I)^{-1}_{(X,\|\cdot\|)\to(\mathrm{D}(S),\|\cdot\|)} \\
&\qquad = I_{(\mathrm{D}(S),\|\!|\cdot|\!\|)\to(\mathrm{D}(S),\|\cdot\|)}(S-\lambda I)^{-1}_{(X,\|\!|\cdot|\!\|)\to(\mathrm{D}(S),\|\!|\cdot|\!\|)}
\end{aligned}$$

is compact being the product of a bounded and a compact operators. □

Theorem 2.6. *Let S be an unbounded non-negative self-adjoint linear operator on a Silbert space X. Then the following are equivalent.*

(i) $\mathrm{EssSpec}(S) = \emptyset$ *(i.e. $\sigma(S)$ is discrete).*

(ii) S has compact resolvent.

(iii) There exists an orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ for X of eigenvectors ϕ_n of the operator S with the corresponding eigenvalues $\lambda_n \ge 0$ such that

$$\lim_{n\to\infty} \lambda_n = \infty.$$

(For a proof see [20, Corollary 4.2.3, p. 77].)

Let S be an unbounded non-negative self-adjoint linear operator with compact resolvent on a Hilbert space X. By the above theorem there exists a countable number of isolated eigenvalues $\lambda_n \ge 0$ of finite multiplicity. This allows to arrange them in non-decreasing order repeating them as many time as their multiplicity,

$$0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \ldots .$$

Moreover

$$\lim_{n\to\infty} \lambda_n = \infty$$

and there exists an orthonormal basis for X of eigenvectors ϕ_n corresponding to the eigenvalues λ_n. So $\phi_n \in \mathrm{D}(S)$, $S\phi_n = \lambda_n\phi_n$, $\|\phi_n\| = 1$, $(\phi_m, \phi_n) = 0$ if $m, n \in \mathbb{N}$ and $m \ne n$, and, for all $u \in X$,

$$u = \sum_{n=1}^{\infty} (u, \phi_n)\phi_n\,.$$

In the sequel, given an unbounded non-negative self-adjoint linear operator S with compact resolvent, we always assume that λ_n and ϕ_n have the meaning described above.

Theorem 2.7. *Let S be an unbounded non-negative self-adjoint linear operator on an Hilbert space X. Then the following statements are equivalent.*

(i) S has compact resolvent.

(ii) $S^{1/2}$ has compact resolvent.

Proof. *(i)⇒(ii)* By Theorem 2.6 there exists an orthonormal basis for X of eigenvectors $\{\phi_n\}_{n\in\mathbb{N}}$ of the operator S with the corresponding eigenvalues $\lambda_n \geq 0$ satisfying $\lim_{n\to\infty} \lambda_n = \infty$. By Lemma 2.3 ϕ_n are also eigenvectors of $S^{1/2}$ with eigenvalues $\lambda_n^{1/2}$ satisfying $\lim_{n\to\infty} \lambda_n^{1/2} = \infty$. Hence by Theorem 2.6 $S^{1/2}$ has compact resolvent. The proof of *(ii)⇒(i)* is similar. □

Lemma 2.3. *Let S be an unbounded non-negative self-adjoint linear operator with compact resolvent on an Silbert space X. Then*

$$\mathrm{D}(S) = \left\{ u \in X \;:\; \sum_{n=1}^{\infty} \lambda_n^2 |(u, \phi_n)|^2 < \infty \right\}$$

and

$$Su = \sum_{n=1}^{\infty} \lambda_n (u, \phi_n)\phi_n$$

for all $u \in \mathrm{D}(S)$, hence for all $u \in \mathrm{D}(S)$, $v \in X$

$$(Su, v) = \sum_{n=1}^{\infty} \lambda_n (u, \phi_n)\overline{(v, \phi_n)}\,.$$

Proof. Since $S : \mathrm{D}(S) \to X$ and $\{\phi_n\}_{n\in\mathbb{N}}$ is a basis for X, for all $u \in \mathrm{D}(S)$ we have

$$Su = \sum_{n=1}^{\infty} (Su, \phi_n)\phi_n = \sum_{n=1}^{\infty} (u, S\phi_n)\phi_n = \sum_{n=1}^{\infty} \lambda_n (u, \phi_n)\phi_n$$

and

$$\|Su\| = \left(\sum_{n=1}^{\infty} \lambda_n^2 |(u, \phi_n)|^2 \right)^{1/2} < \infty\,.$$

Now we prove the inverse. Assume that $u \in X$ and $\sum_{n=1}^{\infty} \lambda_n^2 |(u, \phi_n)|^2 < \infty$. Then the series $\sum_{n=1}^{\infty} \lambda_n (u, \phi_n)\phi_n$ converges in X, say to $v \in X$. Hence

$$\sum_{n=1}^{m} (u, \phi_n)\phi_n \to u \quad \text{in } X$$

and

$$S\left(\sum_{n=1}^{m} (u, \phi_n)\phi_n \right) = \sum_{n=1}^{m} (u, \phi_n) S\phi_n = \sum_{n=1}^{m} \lambda_n (u, \phi_n)\phi_n \to v \quad \text{in } X$$

as $m \to \infty$. Since, being self-adjoint, S is closed it follows that $u \in \mathrm{D}(S)$ (and $Su = v$.) □

2.4 Classes of open sets

For all $E \subset \mathbb{R}^n$ and $\rho > 0$ we set $E_\rho = \{x \in E : \operatorname{dist}(x, \partial E) > \rho\}$; then we recall the following definition.

Let $\rho > 0$, s, $s' \in \mathbb{N}$, $s' \leq s$ and let $\{V_j\}_{j=1}^s$ be a family of bounded open cuboids and $\{r_j\}_{j=1}^s$ be a family of rotations. We say that a bounded open set Ω in $\mathbb{R}^n$ has a continuous boundary with the parameters $\rho, s, s', \{V_j\}_{j=1}^s$, $\{r_j\}_{j=1}^s$, briefly Ω is of class $C(\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ if

(i) $\Omega \subset \cup_{j=1}^s (V_j)_\rho$ and $(V_j)_\rho \cap \Omega \neq \emptyset$ for all $j = 1, ..., s$;

(ii) $V_j \cap \partial\Omega \neq \emptyset$ for all $j = 1, \ldots s'$, and $V_j \subset \Omega_\rho$ for all $s' < j \leq s$;

(iii) there exist real numbers a_{jl}, b_{jl} with $a_{jl} < b_{jl}$ for all $j = 1, \ldots, s$, $l = 1, \ldots, n$ such that

$$r_j(V_j) = \{\, x \in \mathbb{R}^n : \ a_{jl} < x_l < b_{jl}, \text{ for all } l = 1, ..., n\},$$

for all $j = s' + 1, \ldots, s$, and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^n : \ a_{jn} < x_n < g_j(\bar{x}), \ \bar{x} \in W_j\},$$

for all $j = 1, ..., s'$, where $x = (\bar{x}, x_n)$, $\bar{x} = (x_1, ..., x_{n-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{n-1} : \ a_{jl} < x_l < b_{jl}, \text{ for all } l = 1, ..., n-1\}$ and g_j is a uniformly continuous function on W_j; moreover

$$a_{jn} + \rho \leq g_j(\bar{x}) \leq b_{jn} - \rho,$$

for all $j = 1, \ldots, s'$, $\bar{x} \in W_j$.

Sometimes, an entity of the kind $\mathcal{A} = \{\rho, s, s', \{V_j\}_{j=1}^s\}$ is called *an atlas in* $\mathbb{R}^n$ *with parameters* $\rho, s, s', \{V_j\}_{j=1}^s$, or briefly, *an atlas*. We say that an open set Ω is of *class* $C(\mathcal{A})$, or *belongs to the class* $C(\mathcal{A})$, if Ω has a continuous boundary with the parameters $\rho, s, s', \{V_j\}_{j=1}^s$, $\{r_j\}_{j=1}^s$, where $\mathcal{A} = \{\rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\}$.

We say that an open set Ω in $\mathbb{R}^n$ is *an open set with a continuous boundary*, or alternatively *has a continuous boundary*, if Ω is of class $C(\mathcal{A})$ for some atlas $\mathcal{A}$.

Let $\omega : [0, \infty[\to [0, \infty[$ be a continuous increasing function such that $\omega(0) = 0$ and such that, for some $c > 0$

$$\omega(t) \geq ct$$

for all $0 \leq t \leq 1$. We say that a bounded open set having a continuous boundary with the parameters $\rho, s, s', \{V_j\}_{j=1}^s$, $\{r_j\}_{j=1}^s$ is of class $C^{0,\omega(\cdot)}(M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ where $M > 0$ if all the functions g_j satisfy the condition

$$|g_j(\bar{x}) - g_j(\bar{y})| \leq M\omega(|\bar{x} - \bar{y}|),$$

for all $\bar{x}, \bar{y} \in W_j$.

Let $0 < \gamma \leq 1$ and $\omega(a) = a^\gamma$ for all $a \geq 0$. In this case, if Ω is of class $C^{0,\omega(\cdot)}(M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ then Ω has a Hölder continuous boundary and we say that Ω is of class $C^{0,\gamma}(M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ or $\mathrm{Lip}\,(\gamma, M, \rho, s, s', \{V_j\}_{j=1}^s$, $\{r_j\}_{j=1}^s)$.

Furthermore, we say that a bounded open set in $\mathbb{R}^n$ is of class $C^{1,\gamma}(M, \rho, s, s'$, $\{V_j\}_{j=1}^s$, $\{r_j\}_{j=1}^s)$ if all the functions g_j are differentiable and satisfy the condition

$$\left|\frac{\partial g_j}{\partial x_i}(\bar{x})\right| \leq M \quad \text{and} \quad \left|\frac{\partial g_j}{\partial x_i}(\bar{x}) - \frac{\partial g_j}{\partial x_i}(\bar{y})\right| \leq M|\bar{x} - \bar{y}|^\gamma,$$

for all $\bar{x}, \bar{y} \in W_j$, $i = 1, \ldots, n-1$.

Finally if $l \in \mathbb{N}$, then we say that Ω is of class $C^l(M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ if all the functions g_j satisfy

$$\sup_{1 \le |\alpha| \le l} \sup_{\bar{x} \in W_j} |D^\alpha g_j(\bar{x})| \le M.$$

We also say that Ω is of class $C^{0,\omega(\cdot)}$, $C^{0,\gamma}$ *etc.* if it is of class $C^{0,\omega(\cdot)}(M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$, $C^{0,\gamma}(M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ *etc.* for some parameters $M, \rho, s, s', \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s$.

2.5 The weak Laplacian and the Laplacian graph space in $L^2(\Omega)$

The reader of this paper is presumed to be acquainted with the notion of weak (or generalized, or distributional) derivatives. What is needed to us (and much more) can be found, for istance, in [9].

In the same spirit, we define the notion of the weak Laplacian Δ_w.

Definition 2.7. *Let Ω be an open set in $\mathbb{R}^n$, $u, v \in L^1_{\text{loc}}(\Omega)$. The function v is a* weak Laplacian *of u on Ω (briefly $v = \Delta_w u$) if for all $\phi \in C_c^\infty(\Omega)$*

$$\int_\Omega u \Delta\phi \, dx = \int_\Omega v\phi \, dx \, .$$

As in the case of the weak derivatives this definition is equivalent to the following one (the proof is similar.)

Definition 2.8. *Let Ω be an open set in $\mathbb{R}^n$, $u, v \in L^1_{\text{loc}}(\Omega)$. The function v is the* weak Laplacian *of u on Ω (briefly $v = \Delta_w u$) if there exist $\psi_k \in C^\infty(\Omega)$, $k \in \mathbb{N}$, such that*

$$\psi_k \to u, \quad \Delta\psi_k \to v \quad \text{in } L^1_{\text{loc}}(\Omega)$$

as $k \to \infty$.

Remark 2.2. *If u, $\Delta_w u \in L^p_{\text{loc}}(\Omega)$ for some $1 < p < \infty$ then there exist $\psi_k \in C_c^\infty(\Omega)$, $k \in \mathbb{N}$, such that*

$$\psi_k \to u, \quad \Delta\psi_k \to \Delta_w u \quad \text{in } L^p_{\text{loc}}(\Omega)\, .$$

Lemma 2.4. *Let Ω be an open set in $\mathbb{R}^n$ and*

$$\begin{aligned} \mathrm{D}(\Delta_w) &= \left\{ u \in L^1_{\text{loc}}(\Omega) \,:\, \text{there exists } \Delta_w u \text{ on } \Omega \right\} \\ &= \left\{ u \in L^1_{\text{loc}}(\Omega) \,:\, \Delta_w u \in L^1_{\text{loc}}(\Omega) \right\} \, . \end{aligned}$$

Then the weak Laplacian

$$\Delta_w : \mathrm{D}(\Delta_w) \to L^1_{\text{loc}}(\Omega)$$

is the closure of the (ordinary) Laplacian [14]

$$\Delta : C_c^\infty(\Omega) \to L^1_{\text{loc}}(\Omega)\, .$$

[14] The statement also holds if the domain of Δ is assumed to be $C^\infty(\Omega)$ or $C^2(\Omega)$.

Corollary 2.3. *Let Ω be an open set in $\mathbb{R}^n$ and*

$$\mathrm{D}(\Delta_w) = \left\{ u \in L^2(\Omega) \,:\, \Delta_w u \in L^2(\Omega) \right\} .$$

Then the weak Laplacian

$$\Delta_w : \mathrm{D}(\Delta_w) \to L^2(\Omega)$$

is closed.

Lemma 2.4 and Corollary 2.3 are proved similarly to analogous results regarding weak derivatives.

If the weak derivatives $\left(\frac{\partial^2 u}{\partial x_j^2}\right)_w$, $j = 1, \dots, N$, exist on Ω then their own definition and Definition 2.7 immediately imply that $\Delta_w u$ also exists on Ω and

$$\Delta_w u = \sum_{j=1}^{N} \left(\frac{\partial^2 u}{\partial x_j^2}\right)_w . \tag{2.3}$$

However, a natural question arises: if the weak Laplacian $\Delta_w u$ exists on Ω, does it follow that the weak derivatives $\left(\frac{\partial^2 u}{\partial x_j^2}\right)_w$, $j = 1, \dots, N$, exist on Ω and hence equality (2.3) holds? The answer to this question is positive as a consequence of the Sobolev-regularity theory for weak solutions of elliptic equations. For a related result see Theorem 2.9 below.

From now and on we will drop the subscript w from Δ_w used to indicate the weak Laplacian.

As already said, one of the operators which we are mostly concerned about is the Dirichlet Laplacian. Hence we introduce here some notations.

Definition 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then the* Laplacian graph space *$H^{\triangle}(\Omega)$ is defined by*

$$H^{\Delta}(\Omega) = \{u \in L^2(\Omega) \,:\, \Delta u \in L^2(\Omega)\}.$$

On $H^{\Delta}(\Omega)$ we define the inner product

$$(u, v)_{H^{\Delta}(\Omega)} = (u, v)_{L^2(\Omega)} + (\Delta u, \Delta v)_{L^2(\Omega)}$$

for all $u, v \in H^{\Delta}(\Omega)$ and hence, the graph norm

$$\|u\|_{H^{\Delta}(\Omega)} = (\|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)})^{\frac{1}{2}}$$

for all $u \in H^{\Delta}(\Omega)$.

The Laplacian is, as already stated in Corollary 2.3, closed in $L^2(\Omega)$ which is equivalent to say that $H^{\Delta}(\Omega)$ is a Hilbert space.

Our next aim is to study some properties of the weak Laplacian Δ and the space $H^{\Delta}(\Omega)$ generated by it.

We start by noting that $H^2(\Omega) \subset H^{\Delta}(\Omega)$ but in general $H^2(\Omega) \neq H^{\Delta}(\Omega)$.

Lemma 2.5. *Let $n = 2$ or $n = 3$, Ω be an open set in $\mathbb{R}^n$, and $\Omega \neq \mathbb{R}^n$. Then $H^{\Delta}(\Omega) \not\subset H^2(\Omega)$ and $H^{\Delta}(\Omega) \not\subset H^1(\Omega)$.*

Proof. Let $n = 3$. Consider an arbitrary point $y \in \Omega$ and let $r = \mathrm{dist}(y, \partial\Omega)$. Then $B(y,r) \subset \Omega$ and there exists $z \in \partial\Omega \cap \partial B(y,r)$.

First let Ω be bounded. Consider the function $u(x) = -\log|x-z|$ if $n = 2$ and $u(x) = 1/|x-z|$ if $n = 3$. Then $u \in C^\infty(\Omega) \cap L^2(\Omega)$ and $\Delta u = \Delta u = 0$ on Ω, hence $u \in H^\Delta(\Omega)$. On the other hand

$$\frac{\partial u}{\partial x_j} = -\frac{x_j - z_j}{|x-z|^n}\,, \quad \frac{\partial^2 u}{\partial x_j^2} = -\frac{1}{|x-z|^n} + n\frac{(x_j - z_j)^2}{|x-z|^{n+2}} \quad \notin L^2(\Omega)\,,$$

hence $u \notin H^1(\Omega)$ and $u \notin H^2(\Omega)$.

If Ω is unbounded, then let $\eta \in C^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$, $\eta(x) = 1$ if $|x-z| \le 1$ and $\eta(x) = 0$ if $|x-z| \ge 2$. Consider the function $u(x) = \eta(x)/|x-z|$. Then again $u \in C^\infty(\Omega) \cap H^\Delta(\Omega)$ but $u \notin H^1(\Omega)$ and $u \notin H^2(\Omega)$. $\square$

Lemma 2.6. *Let $n \ge 4$ and Ω be an open set in $\mathbb{R}^n$ which is not dense in $\mathbb{R}^n$. Moreover assume that for all $z \in \partial\Omega$ there exists $c = c(z) > 0$ such that*

$$|\Omega \cap B(z,r)| > cr^n$$

for all $0 < r \le 1$. Then $H^\Delta(\Omega) \not\subset H^2(\Omega)$ and $H^\Delta(\Omega) \not\subset H^1(\Omega)$.

Proof. We prove that $H^\Delta(\Omega) \not\subset H^2(\Omega)$, the proof of $H^\Delta(\Omega) \not\subset H^1(\Omega)$ being similar. Since the embedding $I : H^2(\Omega) \to H^\Delta(\Omega)$ is bounded, if we assume that $H^2(\Omega) = H^\Delta(\Omega)$ then, by the Open Map Theorem, I is an homeomorphism of Banach spaces. In particular there exists $C > 0$ such that

$$\|u\|_{H^2(\Omega)} \le C\|u\|_{H^\Delta(\Omega)} \tag{2.4}$$

for all $u \in H^2(\Omega)$. We prove that this is not possible.
Let $y \in \mathbb{R}^n \setminus \overline{\Omega}$, $r = \mathrm{dist(y,\Omega)}$ and $z \in \partial\Omega$ such that $\mathrm{dist(y,z)} = r$. Then set, for every $k \in \ltimes$, $z_k = z + (1/k)(y-z)$. We note that $\{z_k\}_{k\in\ltimes}$ is a sequence in $\mathbb{R}^n \setminus \overline{\Omega}$ which converges to z and such that $\mathrm{dist(z_k,z)} = \mathrm{dist(z_k,\Omega)} = 1/k$. Finally, for every $k \in \ltimes$, we define the functions u_k on Ω by $u_k(x) = 1/|x-z_k|^{n-2}$ and we observe that $u_k \in C^\infty(\Omega) \cap H^2(\Omega)$, $\Delta u_k = 0$ and, for $j = 1, \dots, n$,

$$\frac{\partial^2 u_k}{\partial x_j^2}(x) = (2-n)\frac{1}{|x-z_k|^n} - n(2-n)\frac{(x_j - (z_k)_j)^2}{|x-z_k|^{n+2}}\,.$$

Hence

$$\left|\frac{\partial^2 u_k}{\partial x_j^2}(x)\right|^2 \ge (n-1)(n-2)\frac{1}{|x-z_k|^{2n}}\,,$$

and thus one can prove that there exists $A > 0$ such that

$$\left|\frac{\partial^2 u_k}{\partial x_j^2}(x)\right|^2 \ge \frac{A}{|x-z|^{2n} + |z-z_k|^{2n}} = \frac{A}{|x-z|^{2n} + (1/k)^{2n}}\,.$$

So, if $|x-z| < 1/k$

$$\left|\frac{\partial^2 u_k}{\partial x_j^2}(x)\right|^2 \ge 2A\,k^{2n}\,,$$

which implies

$$\left\|\frac{\partial^2 u_k}{\partial x_j^2}\right\|^2_{L^2(\Omega)} \ge \int_{\Omega\cap B(z,1/k)} \left|\frac{\partial^2 u_k}{\partial x_j^2}(x)\right|^2 dx \ge (2Ac^{-1})\,k^n\,.$$

now let Ω be a bounded. Since $\{z_k\}_{k\in\ltimes}$ is also bounded (every convergent sequence in $\mathbb{R}^n$ is bounded), there exists $R>0$ such that

$$\|u_k\|^2_{L^2(\Omega)} = \int_\Omega \frac{1}{|x-z_k|^{2n-4}}\,dx \le \int_{B(z_k,R)\setminus B(z_k,1/k)} \frac{1}{|x-z_k|^{2n-4}}\,dx\,.$$

Then there exists a constant $A'>0$ depending only on n such that, for $n>4$,

$$\|u_k\|^2_{L^2(\Omega)} \le A'\,k^{n-4}\,,$$

and for $n=4$

$$\|u_k\|^2_{L^2(\Omega)} \le A'\,\log\,k\,.$$

Then, for $n>4$, we have

$$\frac{\|u_k\|^2_{H^2(\Omega)}}{\|u_k\|^2_{H^\Delta(\Omega)}} \ge \frac{\|u_k\|^2_{L^2(\Omega)} + \left\|\frac{\partial u_k}{\partial x_j^2}\right\|^2_{L^2(\Omega)}}{\|u_k\|^2_{L^2(\Omega)}} \ge 1 + A''\,k^4\,, \tag{2.5}$$

and, for $n=4$,

$$\frac{\|u_k\|^2_{H^2(\Omega)}}{\|u_k\|^2_{H^\Delta(\Omega)}} \ge 1 + A''\,k^n(\log\,k)^{-1} \tag{2.6}$$

(here $A''=2A/A'$.) Both the right-hand side of (2.5) and (2.6) tend to $+\infty$ as $k\to+\infty$. This is in contradiction with (2.4) and thus it is not possible that $H^2(\Omega)=H^\Delta(\Omega)$.

If Ω is not bounded we consider the function $\eta\in C^\infty(\mathbb{R}^n)$ introduced in the proof of the previous lemma ($0\le\eta\le1$, $\eta(B(z,1))=\{1\}$, $\operatorname{supp}\eta\subset B(z,2)$) and we define, for every $k\in\ltimes$, $u_k(x)=\eta(x)/|x-z_k|^{n-2}$. Then again we prove that $\|u_k\|_{H^2(\Omega)}/\|u_k\|_{H^\Delta(\Omega)}$ is not bounded and thus $H^\Delta(\Omega)\not\subset H^2(\Omega)$. $\square$

Remark 2.3. *It is well-known (cf. [9, Chapter 6]) that for any open set Ω in $\mathbb{R}^n$ of class $C^{0,1}$ there exists a bounded linear extension operator* [15]

$$E: H^2(\Omega)\to H^2(\mathbb{R}^n).$$

However, this is not the case for the spaces $H^\Delta(\Omega)$. Moreover for any open sets Ω in $\mathbb{R}^n$ with $n\ge2$ satisfying for $n=2,3$ the assumptions of Lemma 2.5 and for $n\ge4$ the assumptions of Lemma 2.6 there does not exist an extension operator

$$E: H^\Delta(\Omega)\to H^\Delta(\mathbb{R}^n).$$

Indeed if it exists, then $Eu\in H^\Delta(\mathbb{R}^n)$ for all $u\in H^\Delta(\Omega)$, and since $H^\Delta(\mathbb{R}^n)=H^2(\mathbb{R}^n)$ as a consequence of the Sobolev-regularity theory for solutions of Poisson equation, it would follow that $Eu\in H^2(\mathbb{R}^n)$ which implies that $u\in H^2(\Omega)$ and $H^\Delta(\Omega)\subset H^2(\Omega)$ and this contradicts Lemmas 2.5, 2.6.

[15] I.e., $(Eu)(x)=u(x)$ for all $x\in\Omega$.

2.6 The Dirichlet Laplacian operator

For an open set $\Omega \in \mathbb{R}^n$, the Dirichlet Laplacian is usually defined via the Friedrich's extension procedure.

Theorem 2.8. *Let Ω be an open set in $\mathbb{R}^n$, ($n \in \mathbb{N}$), such that the imbedding $H_0^1(\Omega) \subset L^2(\Omega)$ is compact. Then there exists a non-negative self-adjoint operator $S^F := S_\Omega^F$ in $L^2(\Omega)$, called* the (generalized) minus Dirichlet Laplacian relative to the domain Ω, *with compact resolvent such that.*

$$\mathrm{D}({S_\Omega^F}^{\frac{1}{2}}) = H_0^1(\Omega),$$

$$\begin{aligned}\mathrm{D}(S_\Omega^F) &= \{u \in H_0^1(\Omega) \ : \ {S_\Omega^F}^{\frac{1}{2}} u \in H_0^1(\Omega)\} \\ &= H_0^1(\Omega) \cap H^\Delta(\Omega),\end{aligned}$$

$$S_\Omega^F u = -\Delta u, \qquad u \in \mathrm{D}(S_\Omega^F),$$

$$\int_\Omega {S_\Omega^F}^{\frac{1}{2}} u \overline{{S_\Omega^F}^{\frac{1}{2}} v} dx = \int_\Omega \nabla u \cdot \overline{\nabla v} dx$$

for all $u, v \in H_0^1(\Omega)$,

$$\int_\Omega S_\Omega^F u \overline{v} dx = \int_\Omega \nabla u \cdot \overline{\nabla v} dx$$

for all $u \in \mathrm{D}(S_\Omega^F)$ and $v \in H_0^1(\Omega)$).

Moreover, $\lambda \in \mathbb{C}$ is an eigenvalue of S_Ω^F and $u \in \mathrm{D}(S_\Omega^F)$, $u \neq 0$, a corresponding eigenfunction if, and only if,

$$\int_\Omega \nabla u \cdot \overline{\nabla v} dx = \lambda \int_\Omega u \overline{v} dx$$

for all $v \in H_0^1(\Omega)$. Thus, in particular, $\lambda \in \mathbb{R}$ and $\lambda > 0$.

The operator S_Ω^F has compact resolvent, hence its spectrum is discrete, that is to say, it consists only of isolated eigenvalues of finite multiplicity. If we arrange these eigenvalues in non-decreasing order repeating them as many times as their multiplicity

$$0 < \lambda_1[S_\Omega^F] \leq \lambda_2[S_\Omega^F] \cdots \leq \lambda_2[S_\Omega^F] \leq \cdots,$$

then

$$\lim_{n \to \infty} \lambda_n[S_\Omega^F] = \infty.$$

Finally, $L^2(\Omega)$ admits an orthonormal basis $\{\varphi_n[S_\Omega^F]\}_{n \in \mathbb{N}}$ consisting of eigenfunctions of S_Ω^F (in these notations, the eigenfunction $\varphi_n[S_\Omega^F]$ is chosen to correspond to the eigenvalue $\lambda_n[S_\Omega^F]$, $n \in \mathbb{N}$).

Henceforth, we shall use alternatively the notation $-\Delta_{\Omega,\mathrm{D}}$ to indicate the generalized minus Dirichlet Laplacian in the open set Ω.

As to the question of the (global) Sobolev-regularity exhibited by functions belonging to the domain of $-\Delta_{\Omega,\mathrm{D}}$, we have the following result.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that $\Omega \in C^{1,r}$ with some $\frac{1}{2} < r < 1$.) Then*

$$\mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) \subset H^2(\Omega).$$

In particular,

$$\mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) = H_0^1(\Omega) \cap H^\Delta(\Omega) = H_0^1(\Omega) \cap H^2(\Omega). \tag{2.7}$$

Proof. If $\Omega \in C^{1,1}$ this inclusion is a consequence of the a priori L^p-estimates for solutions to elliptic equations which can be considered a classical result by now and can be found, for instance, in [31, Theorem 9.15, §9.5, p. 239]. The more general case of $\Omega \in C^{1,r}$ with $\frac{1}{2} < r < 1$ see [27, Lemma A.1, p. 32]. $\square$

Theorem 2.10. *Let $n = 1, 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $\Omega \in C^{1,r}$ with $\frac{1}{2} < r < 1$. Then*

$$\mathrm{D}(-\Delta_{\Omega,\mathrm{D}}) = H_0^1(\Omega) \cap H^2(\Omega) = \overline{\{u \in C^2(\Omega) \cap C(\overline{\Omega}) \cap H^2(\Omega) \ : \ u|_{\partial\Omega} = 0\}}, \tag{2.8}$$

where the closure is intended in $H^2(\Omega)$.

Proof. The first equality is explained in the previous theorem.

To prove the second equality, we note that, since $n \leq 3$, by the Sobolev embedding theorem each function $u \in H_0^1(\Omega) \cap H^2(\Omega)$ is equivalent to a function g belonging to $C(\overline{\Omega})$. Since $u \in H_0^1(\Omega)$ it has zero trace at $\partial\Omega$, for example, in the sense of [9, Ch.5]. Therefore $g|_{\partial\Omega} = 0$.

By Theorem 3 and Corollary 2 of Ch.2 of [9] it follows that there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega) \cap C(\overline{\Omega}) \cap H^2(\Omega)$ such that $g_k \to u$ in $H^2(\Omega)$ and $g_k \to g$ in $C(\overline{\Omega})$ as $k \to \infty$ and $g_k|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$. If fact for any function $\mu \in C(\Omega)$ there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega) \cap H^2(\Omega)$ such that

$$\lim_{k\to\infty} \|(D^\alpha u - D^\alpha g_k)\mu\|_{L^2(\Omega)}$$

for all multi-indices $|\alpha| \leq 2$. Actually, the construction of these functions $\{g_k\}_{k\in\mathbb{N}}$ is independent of the summability parameter p of the Sobolev space $W^{1,p}(\Omega)$ and, for all $1 \leq p < \infty$ and for all multi-indices $|\alpha| \leq 2$

$$\lim_{k\to\infty} \|(D^\alpha u - D^\alpha g_k)\mu\|_{L^2(\Omega)}$$

and

$$\lim_{k\to\infty} \|(g - g_k)\mu\|_{L^\infty(\Omega)} = 0.$$

If we take $\mu(x) = (\mathrm{dist}(x, \partial\Omega))^{-1}$, $x \in \Omega$, we derive that $\|g_k - u\|_{H^2(\Omega)} \to 0$ as $k \to \infty$ and that, as in the proof of Corollary 2 in [9, Ch.2], $g_k|_{\partial\Omega} = 0$. Moreover, $g_k \to g$ in $C(\overline{\Omega})$ as $k \to \infty$. $\square$

2.7 Direct sum of operators

In this subsection let X, X' indicate normed spaces: any subspace $M \subset X$ can be seen as subspace of $X \times X'$ by identifying it with

$$\{(u, 0) \ : \ u \in M\};$$

analogously can be done for any subspace $M' \subset X'$ by identifying it with the subspace

$$\{(0, u') \ : \ u' \in M'\}$$

of $X \times X'$.

Hence, in this notation we have $X \times X' = X \dot{+} X'$, where the binary operator $\dot{+}$ indicates the direct sum between subspaces of a vector space (In the case of

Hilbert spaces, if the subspaces are orthogonal to each other, we will use the symbol $\oplus$ instead of $\dotplus$ to denote the relative direct sum).

If $u \in X$ and $v \in X'$, as we said, we can see u and v as elements of $X \dotplus X'$ by identifying them with $(u, 0)$ and $(0, v)$ respectively; in this context we have

$$(u, v) = u + v.$$

Definition 2.10. *Let be given two linear operators $S \in \mathcal{L}(X, Y)$, $S' \in \mathcal{L}(X', Y')$, where X, X', Y, Y' are normed spaces. Then we define the* direct sum operator *$S \dotplus S' \in \mathcal{L}(X \dotplus X', Y \dotplus Y')$ in the following way:*

$$\mathrm{D}(S \dotplus S') = \mathrm{D}(S) \dotplus \mathrm{D}(S'),$$

and

$$(S \dotplus S')(u + u') = Su + S'u'$$

for all $u \in \mathrm{D}(S)$, $u' \in \mathrm{D}(S')$.

Remark 2.4. *If X, X', Y, Y' are Hilbert spaces we shall use the notation $S \oplus S'$ instead of $S \dotplus S'$. Thus, $S \oplus S' \in \mathcal{L}(X \oplus X',\ Y \oplus Y')$.*

A list of easily verifiable properties of the direct sum operator is promptly given.

Proposition 2.1. *Given certain normed spaces X, X', Y, Y', let $S \in \mathcal{L}(X, Y)$ and $S' \in \mathcal{L}(X', Y')$ be linear operators. Then we have:*

(i)
$$\mathrm{N}(S \dotplus S') = \mathrm{N}(S) \dotplus \mathrm{N}(S'), \qquad R(S \dotplus S') = R(S) \dotplus R(S');$$

(ii)
$$G(S \dotplus S') = G(S) \dotplus G(S');$$

(iii) $S \dotplus S'$ is invertible if, and only if, S, S' are both invertible and, in this case,
$$(S \dotplus S')^{-1} = S^{-1} \dotplus S'^{-1};$$

(iv) if S and S' are closable operators, then $S \dotplus S'$ is also a closable operator and
$$\overline{S \dotplus S'} = \bar{S} \dotplus \bar{S}';$$
in particular, if S, S' are closed operators, then so is $S \dotplus S'$;

(v) if S and S' are densely defined operators, then $S \dotplus S'$ is a densely defined operator too and
$$(S \dotplus S')^* = S^* \dotplus S'^*;$$

(vi) if $X = Y$, $X' = Y'$ are Hilbert spaces and S, S' are (closable) densely defined operators, then $S \oplus S'$ is self-adjoint (essentially self-adjoint) if, and only if, both S, S' are self-adjoint (essentially self-adjoint) operators;

(vii) $S \dotplus S'$ is a bounded (respectively, compact) operator if, and only if, both S and S' are bounded (respectively, compact) operators;

(viii) *if* $X = Y$, $X' = Y'$,

$$\sigma(S \dotplus S') = \sigma(S) \cup \sigma(S') \qquad \rho(S \dotplus S') = \rho(S) \cap \rho(S');$$

(ix)

$$\sigma_p(S \oplus S') = \sigma_p(S) \cup \sigma_p(S'),$$

and for any $\lambda \in \mathbb{C}$,

$$\mathrm{N}(S \dotplus S' - \lambda I_{X \dotplus X'}) = \mathrm{N}(S - \lambda I_X) \dotplus \mathrm{N}(S' - \lambda I_{X'}),$$

hence, in particular, the geometrical multiplicity of λ *as an eigenvalue of* $S \dotplus S'$ *is equal to the sum of the geometrical multiplicities of* λ *as an eigenvalue of* S *and* S' *(this assertion is valid with the understanding that if* λ *is not an actual eigenvalue of an operator, then* λ *is looked at as an eigenvalue of geometric multiplicity zero of the said operator);*

(x) *if* $X = Y$, $X' = Y'$ *are Hilbert spaces and* S, S' *are non-negative symmetric densely defined linear operators, then about the Friedreich extensions subsists the result*

$$(S \oplus S')^F = S^F \oplus S'^F;$$

(xi) *if* $X = Y$, $X' = Y'$, *then* $S \oplus S'$ *is an operator with compact resolvent if, and only if,* S, S' *are operators with compact resolvent.*

3 Gaps between operators defined on different open sets

3.1 Preliminary definitions and examples

We shall be interested in the comparison of unbounded closed or closable linear operators acting in different Banach spaces, namely $S : \mathrm{D}(S) \subset X_1 \to Y_1$, $T : \mathrm{D}(T) \subset X_2 \to Y_2$. For this purpose Definition 1.3 should be modified. It is not clear how to do this, but if one concentrates first at the case in which $X_1 = Y_1 = L^2(\Omega_1)$, $X_2 = Y_2 = L^2(\Omega_2)$, where Ω_1, Ω_2 are open set in $\mathbb{R}^n$, then one way that one might think of is to use the intersection $\Omega_1 \cap \Omega_2$ of the open sets where the operators are both defined. After all we are first of all interested on "small" perturbations of the open set Ω, where the operator is defined, hence the intersection $\Omega_1 \cap \Omega_2$ will not be much "distant" or "different" from each of the sets Ω_1 and Ω_2, and it is natural to hope that one might glean information for each of the operators S, T in a certain sense by just looking at the intersection of the open sets where they are defined.

Let us make this idea precise with the following definition.

Definition 3.1. *Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^n$ be open sets and*

$$S : \mathrm{D}(S) \subset L^2(\Omega_1) \to L^2(\Omega_1),$$

$$T : \mathrm{D}(T) \subset L^2(\Omega_2) \to L^2(\Omega_2)$$

linear operators. Then we define

$$\delta_0(S,T) = \sup_{\substack{u \in \mathrm{D}(S) \\ \|u\|^2_{L^2(\Omega_1)} + \|Su\|^2_{L^2(\Omega_1)} = 1}} \inf_{v \in \mathrm{D}(T)} \left(\|u - v\|^2_{L^2(\Omega_1 \cap \Omega_2)} + \|Su - Tv\|^2_{L^2(\Omega_1 \cap \Omega_2)} \right)^{\frac{1}{2}} \tag{3.1}$$

and

$$\tilde{\delta}_0(S,T) = \max[\delta(S,T), \delta(T,S)]. \tag{3.2}$$

In the thesis $\delta_0(S,T)$ and $\hat{\delta}_0(S,T)$ will be called preliminary gap *and* preliminary symmetric gap *between the operators S and T.*

Example 3.1. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, be open sets. We consider the generalized minus Dirichlet Laplacian operators $-\Delta_{\Omega_i,\mathrm{D}}$ associated to the open sets Ω_i $(i = 1, 2)$ respectively as outlined in Theorem 2.8. For the preliminary gap between these two operators we have*

$$\delta_0(-\Delta_{\Omega_1,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}}) = \sup_{\substack{u \in H^1_0(\Omega_1) \cap H^\Delta(\Omega_1) \\ \|u\|_{H^\Delta(\Omega_1)} = 1}} \inf_{v \in H^1_0(\Omega_2) \cap H^\Delta(\Omega_2)} \left(\|u - v\|^2_{L^2(\Omega_1 \cap \Omega_2)} + \|\Delta u - \Delta v\|^2_{L^2(\Omega_1 \cap \Omega_2)} \right)^{\frac{1}{2}}$$

If the boundaries of the open sets Ω_i, $i = 1, 2$, satisfy the regularity assumptions of Theorem 2.9, that is if $\Omega_i \in C^{1,r}$ with $\frac{1}{2} < r < 1$ then in the previous expression the space $H^\Delta(\Omega_i)$ can be replaced with the Sobolev space $H^2(\Omega_i)$ $(i = 1, 2)$.

If moreover, the dimension of the space $n \leq 3$ so that the approximation Theorem 2.10 holds, then

$$\delta_0(-\Delta_{\Omega_1,\mathrm{D}},\, -\Delta_{\Omega_1,\mathrm{D}}) = \sup_{\substack{u \in C^2(\Omega_1)\cap C(\overline{\Omega_1}) \\ \cap H^2(\Omega_1),\ u|_{\partial\Omega_1}=0 \\ \|u\|^2_{L^2(\Omega_1)} + \|\Delta u\|^2_{L^2(\Omega_1)} \\ =1}} \inf_{\substack{v \in C^2(\Omega_2)\cap C(\overline{\Omega_2}) \\ \cap H^2(\Omega_2) \\ v|_{\partial\Omega_2}=0}} \Big(\|u-v\|^2_{L^2(\Omega_1\cap\Omega_2)} + \|\Delta u - \Delta v\|^2_{L^2(\Omega_1\cap\Omega_2)}\Big)^{\frac{1}{2}}.$$

Example 3.2. *In Example 3.1 let, $\Omega_1 = I :=]0,1[$, $\Omega_2 = I_\varepsilon :=]0,1+\varepsilon[$, for any fixed $\varepsilon > 0$. In this case $-\Delta_{\Omega_2,D}$ depends on a parameter ε and we now denote it for brevity by S_ε. Also S will stand for $-\Delta_{\Omega_1,D}$. Note that in the one dimensional case $H^\Delta(I) = H^2(I)$ for any open interval I. Thus*

$$\mathrm{D}(S) = \{u \in C^2(0,1) \cap C([0,1]) \cap H^2(0,1) : u(0) = u(1) = 0\}$$

and

$$\mathrm{D}(S_\varepsilon) = \{u \in C^2(0,1+\varepsilon) \cap C([0,1+\varepsilon]) \cap H^2(0,1+\varepsilon) : u(0) = u(1+\varepsilon) = 0\}.$$

Regarding the gap we have

$$\delta_0(S, S_\varepsilon) = 0$$

because for any $u \in \mathrm{D}(S) \subset H^2(0,1)$ there exists $\tilde{u} \in H^2(\mathbb{R})$ such that $\tilde{u}(x) = u(x)$ for all $x \in (0,1)$ (see [9], for example Section 6.1). Moreover, the extension function $\tilde{u}$ can be chosen in such a way that $\tilde{u}(1+\varepsilon) = \tilde{u}'(1+\varepsilon) = 0$, that is, $\tilde{u} \in H^2_0(0,1+\varepsilon)$. But $C^\infty_c(0,1+\varepsilon)$, and consequently $\mathrm{D}(S_\varepsilon)$ which contains the former, is dense in $H^2_0(0,1+\varepsilon)$. Hence

$$\mathrm{dist}_{H^2(0,1)}(u, \mathrm{D}(S_\varepsilon)) \leq \mathrm{dist}_{H^2(0,1+\varepsilon)}(\tilde{u}, \mathrm{D}(S_\varepsilon)) = 0.$$

Therefore, taking the supremum over all of $u \in \mathrm{D}(S)$ with $\|u\|_{H^2(0,1)} = 1$, we conclude that $\delta_0(S, S_\varepsilon) = 0$.

On the other hand $\delta_0(S_\varepsilon, S)$ is more complex to be evaluated, and on this regard, we have the following proposition.

Proposition 3.1. *There exists a real number $B > 0$ such that*

$$\delta_0(S_\varepsilon, S) = B\varepsilon + O(\varepsilon^2) \tag{3.3}$$

as $\varepsilon \to 0^+$.

For the proof of this proposition, we need the following lemmas.

Lemma 3.1. *There exists $A > 0$ such that*

$$\inf_{\substack{g \in C^2(I_\varepsilon)\cap C(\overline{I_\varepsilon}) \\ g(0)=g(1+\varepsilon)=0 \\ g(1)=1}} \int_0^{1+\varepsilon} (g^2 + (g'')^2)dx = \frac{A}{\varepsilon^2}\big(1 + O(\varepsilon)\big) \tag{3.4}$$

as $\varepsilon \to 0^+$.

Proof. We are dealing with a calculus of variations problem, and as we know, a solution $g \in C^4([0, 1+\varepsilon])$ of such a problem satisfies the corresponding Euler-Lagrange equation:

$$g^{(IV)} + g = 0. \tag{3.5}$$

The set of the solutions of this equation is a four-dimensional vector space, a basis of which is constituted by the following four functions:

$$\sinh\left(\frac{x}{\sqrt{2}}\right)\sin\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right),$$

$$\sinh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right),$$

$$\cosh\left(\frac{x}{\sqrt{2}}\right)\cos\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)$$

and

$$\cosh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\cos\left(\frac{x}{\sqrt{2}}\right)$$

Thus, the generic solution of (3.5) g is given by a linear combination of these functions:

$$\begin{aligned} g(x) \quad = \quad & c_1 \sinh\left(\frac{x}{\sqrt{2}}\right)\sin\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right) + c_2 \sinh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right) \\ & + c_3 \cosh\left(\frac{x}{\sqrt{2}}\right)\cos\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right) + c_4 \cosh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\cos\left(\frac{x}{\sqrt{2}}\right), \end{aligned}$$

where c_1, c_2, c_3, c_4 are real numbers. Now imposing the conditions $g(0) = g(1+\varepsilon) = 0$, we get a homogeneous linear system of two equations with two unknowns c_1 and c_2, whose solution is $c_3 = c_4 = 0$, since for all $\varepsilon > 0$

$$\begin{vmatrix} \cos\frac{1+\varepsilon}{\sqrt{2}} & \cosh\frac{1+\varepsilon}{\sqrt{2}} \\ \cosh\frac{1+\varepsilon}{\sqrt{2}} & \cos\frac{1+\varepsilon}{\sqrt{2}} \end{vmatrix} \neq 0.$$

Imposing the remaining condition $g(1) = 1$, we can express the coefficient c_2 in terms of the other coefficient c_1 in the following manner:

$$c_2 = -\frac{1 + \sinh\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{\varepsilon}{\sqrt{2}}\right)c_1}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)}.$$

So, for the function g we have the following

$$
\begin{aligned}
g(x) &= c_1 \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right) \\
&\quad - \frac{1+\sinh\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{\varepsilon}{\sqrt{2}}\right)c_1}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)} \sinh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right) \\
&= c_1 \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right) \\
&\quad - c_1 \frac{\sinh\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{\varepsilon}{\sqrt{2}}\right)}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)} \sinh\frac{x-1-\varepsilon}{\sqrt{2}} \sin\frac{x}{\sqrt{2}} \\
&\quad - \frac{\sinh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right)}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)} = c_1\varphi_\epsilon(x) - \psi_\varepsilon(x),
\end{aligned}
$$

where

$$
\varphi_\epsilon(x) = \sinh\left(\frac{x}{\sqrt{2}}\right) \sin\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right) - \frac{\sinh\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{\varepsilon}{\sqrt{2}}\right)}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)} \sinh\frac{x-1-\varepsilon}{\sqrt{2}} \sin\frac{x}{\sqrt{2}}
$$

and

$$
\psi_\varepsilon(x) = \frac{\sinh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right)}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)}.
$$

For later use, we introduce also the following notations. We put

$$
\varphi_0(x) = \lim_{\varepsilon\to 0}\varphi_\varepsilon(x) = \sinh\left(\frac{x}{\sqrt{2}}\right)\sin\left(\frac{x-1}{\sqrt{2}}\right) - \frac{\sinh\left(\frac{1}{\sqrt{2}}\right)}{\sin\left(\frac{1}{\sqrt{2}}\right)} \sinh\frac{x-1}{\sqrt{2}} \sin\frac{x}{\sqrt{2}},
$$

$$
\chi_\varepsilon(x) = \varepsilon\psi_\varepsilon(x) = \varepsilon\frac{\sinh\left(\frac{x-1-\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right)}{\sinh\left(\frac{\varepsilon}{\sqrt{2}}\right)\sin\left(\frac{1}{\sqrt{2}}\right)}
$$

and

$$
\chi_0(x) = \lim_{\varepsilon\to 0}\chi_\varepsilon(x) = \sqrt{2}\frac{\sinh\left(\frac{x-1}{\sqrt{2}}\right)\sin\left(\frac{x}{\sqrt{2}}\right)}{\sin\left(\frac{1}{\sqrt{2}}\right)}.
$$

Till now we have found that the solution of the given minimization problem lies in the one-parameter family $\{c_1\varphi_\epsilon(x) - \psi_\varepsilon(x) : c_1 \in \mathbb{R}\}$. To obtain the minimal solution, we should minimize throughout this family of functions, in other words, we have to solve the following one variable minimization problem

$$
\inf_{c_1\in\mathbb{R}} \int_0^{1+\varepsilon} \left[(c_1\varphi_\epsilon(x) - \psi_\varepsilon(x))^2 + (c_1\varphi''_\epsilon(x) - \psi''_\varepsilon(x))^2\right] dx.
$$

For c_1 to be a minimum point, it must annul the first derivative of the function being minimized, e.i.

$$\int_0^{1+\varepsilon} [(c_1\varphi_\epsilon(x) - \psi_\varepsilon(x))\varphi_\varepsilon(x) + (c_1\varphi''_\epsilon(x) - \psi''_\varepsilon(x))\varphi''_\varepsilon(x)]dx = 0.$$

Thus, we get[16]

$$c_1 = \frac{(\varphi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}}.$$

Therefore, the solution of the problem (3.4) isgiven by the formula

$$g_\varepsilon = \frac{(\varphi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}}\varphi_\varepsilon - \psi_\varepsilon.$$

The minimum is given by the following formula:

$$\begin{aligned}
J(\varepsilon) \quad := \quad & (g_\varepsilon, g_\varepsilon)_{H^2(0,1+\varepsilon)} = \frac{(\varphi_\varepsilon, \psi_\varepsilon)^2_{H^2(0,1+\varepsilon)}}{(\varphi_\varepsilon, \varphi_\varepsilon)^2_{H^2(0,1+\varepsilon)}}(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)} + (\psi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} \\
& -2\frac{(\varphi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}}(\varphi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} \\
= \quad & \frac{1}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}}\Big((\varphi_\varepsilon, \psi_\varepsilon)^2_{H^2(0,1+\varepsilon)} + (\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}(\psi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} \\
& -2(\varphi_\varepsilon, \psi_\varepsilon)^2_{H^2(0,1+\varepsilon)}\Big) \\
= \quad & \frac{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}(\psi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} - (\varphi_\varepsilon, \psi_\varepsilon)^2_{H^2(0,1+\varepsilon)}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}} \\
= \quad & \frac{\begin{vmatrix} (\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)} & (\varphi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} \\ (\varphi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} & (\psi_\varepsilon, \psi_\varepsilon)_{H^2(0,1+\varepsilon)} \end{vmatrix}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}} \\
= \quad & \frac{1}{\varepsilon^2} \cdot \frac{\begin{vmatrix} (\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)} & (\varphi_\varepsilon, \chi_\varepsilon)_{H^2(0,1+\varepsilon)} \\ (\varphi_\varepsilon, \chi_\varepsilon)_{H^2(0,1+\varepsilon)} & (\chi_\varepsilon, \chi_\varepsilon)_{H^2(0,1+\varepsilon)} \end{vmatrix}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}} \\
= \quad & \frac{1}{\varepsilon^2} \cdot \left(\frac{\begin{vmatrix} (\varphi_0, \varphi_0)_{H^2(0,1)} & (\varphi_0, \chi_0)_{H^2(0,1)} \\ (\varphi_0, \chi_0)_{H^2(0,1)} & (\chi_0, \chi_0)_{H^2(0,1)} \end{vmatrix}}{(\varphi_0, \varphi_0)_{H^2(0,1)}} + O(\varepsilon) \right),
\end{aligned}$$

because the function of ε

$$\varepsilon \to \frac{\begin{vmatrix} (\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)} & (\varphi_\varepsilon, \chi_\varepsilon)_{H^2(0,1+\varepsilon)} \\ (\varphi_\varepsilon, \chi_\varepsilon)_{H^2(0,1+\varepsilon)} & (\chi_\varepsilon, \chi_\varepsilon)_{H^2(0,1+\varepsilon)} \end{vmatrix}}{(\varphi_\varepsilon, \varphi_\varepsilon)_{H^2(0,1+\varepsilon)}}$$

is analytic in $\varepsilon = 0$. The above Gram determinants are positive because the functions are linearly independent. □

[16] The definitions are standard: $(f, g)_{H^2(0,1+\varepsilon)} = \int_0^{1+\varepsilon} (f(x)g(x) + f''(x)g''(x))dx$ for any pair of twice continuously differentiable functions f and g on the interval $[0, 1+\varepsilon]$.

To prove Proposition 3.1 we have to estimate

$$\delta_0(S_\varepsilon, S) = \sup_{\substack{g \in C^2(I_\varepsilon)\cap C(\overline{I_\varepsilon}) \\ g(0)=g(1+\varepsilon)=0 \\ \int_0^{1+\varepsilon}(g^2+(g'')^2)dx=1}} \operatorname{dist}_{H^2(0,1)}(g,\ \mathrm{D}(S)), \tag{3.6}$$

where

$$\operatorname{dist}_{H^2(0,1)}(g,\ \mathrm{D}(S)) = \inf_{\substack{f\in C^2(I)\cap C(\overline{I}) \\ f(0)=f(1)=0}} \left(\int_0^1 [(g-f)^2+(g''-f'')^2]dx\right)^{\frac{1}{2}}. \tag{3.7}$$

Hence, the following lemma will be needed.

Lemma 3.2. *There exists a positive number k satisfying the inequality $\frac{1}{4\sqrt{2}} \leq k \leq \frac{1}{\sqrt{3}}$ such that for all functions $g \in C^2(I)\cap C(\overline{I})$ with $g(0)=0$*

$$\operatorname{dist}_{H^2(0,1)}(g,\ \mathrm{D}(S)) = k|g(1)|\,. \tag{3.8}$$

Proof. If we make the transformations $u = g - f$, $v = u/g(1)$, assuming that $g(1) \neq 0$, then we have

$$\begin{aligned}\operatorname{dist}_{H^2(0,1)}(g,\ \mathrm{D}(S)) &= \inf_{\substack{u\in C^2(I)\cap C(\overline{I}) \\ u(0)=0\ u(1)=g(1)}} \left(\int_0^1 [u^2+(u'')^2]dx\right)^{\frac{1}{2}} \\ &= |g(1)| \inf_{\substack{v\in C^2(I)\cap C(\overline{I}) \\ v(0)=0\ v(1)=1}} \left(\int_0^1 [v^2+(v'')^2]dx\right)^{\frac{1}{2}}.\end{aligned}$$

These equalities are also true if $g(1)=0$; in this case just put $f=g$ in (3.7) to obtain $\operatorname{dist}_{H^2(0,1)}(g,\ \mathrm{D}(S))=0$. Thus, we obtain (3.8) with

$$k = \inf_{\substack{v\in C^2(I)\cap C(\overline{I}) \\ v(0)=0\ v(1)=1}} \left(\int_0^1 [v^2+(v'')^2]dx\right)^{\frac{1}{2}}. \tag{3.9}$$

It remains to prove that k satisfies the stated bounds. To estimate k from below we use inequality (3.26), page 88 of [9] in the following way:

$$\begin{aligned} 1 &= |v(1)| = \left|\int_0^1 v'(x)dx\right| \leq \int_0^1 |v'(x)|dx \\ &\leq 4\int_0^1 (|v(x)|+|v''(x)|)dx \\ &\leq 4\sqrt{2}\left(\int_0^1 (|v|^2+|v''|^2)dx\right)^{\frac{1}{2}} \end{aligned}$$

for any function $v \in C^2(I) \cap C(\overline{I})$ such that $v(0) = 0$ and $v(1) = 1$. To estimate k from above it suffices to plug $v(x) = x$ in (3.9). $\square$

Proof of Proposition 3.1. With formula (3.8) and Lemma 3.1 we obtain

$$
\begin{aligned}
\delta_0(S_\varepsilon, S) &= \sup_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0 \\ \int_0^{1+\varepsilon} (g^2 + (g'')^2)dx = 1}} k|g(1)| \\
&= k \cdot \sup_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0}} \frac{|g(1)|}{\sqrt{\int_0^{1+\varepsilon} (g^2 + (g'')^2)dx}} \\
&= \frac{k}{\inf\limits_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0 \\ g(1) = 1}} \sqrt{\int_0^{1+\varepsilon} (g^2 + (g'')^2)dx}} \\
&= \left(\frac{k}{\frac{A}{\varepsilon^2}\big(1 + O(\varepsilon)\big)} \right)^{\frac{1}{2}} \\
&= B\varepsilon\big(1 + O(\varepsilon)\big), \qquad (3.10)
\end{aligned}
$$

as $\varepsilon \to 0^+$, where $B = \sqrt{k/A}$. Thus, the proof is concluded. $\square$

In Proposition 3.1 the exact order of behaviour of $\delta_0(S_\varepsilon, S)$ is established. However for applications it suffices to obtain sharp estimates from above for $\delta_0(S_\varepsilon, S)$ which is a much simpler problem.

Lemma 3.3. *For all* $0 < \varepsilon < 1$

$$
\frac{1}{4\sqrt{2 + \pi^2}}\varepsilon < \delta_0(S_\varepsilon, S) < \frac{8}{\sqrt{3}}\,\varepsilon\,. \qquad (3.11)
$$

Proof. Let $0 < \varepsilon < 1$. Take any $g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon})$ such that $g(0) = g(1+\varepsilon) = 0$ and $\|g\|_{H^2(0,1+\varepsilon)} = 1$. By using again inequality (3.26) of [9], page 88, note that

$$
\begin{aligned}
|g(1)| &= \left| -\int_1^{1+\varepsilon} g'(x)dx \right| \le \varepsilon \|g'\|_{C([0,1+\varepsilon])} \\
&\le 4\varepsilon \left(\frac{1}{(1+\varepsilon)^2} \int_0^{1+\varepsilon} |g(x)|dx + \int_0^{1+\varepsilon} |g''(x)|dx \right) \\
&\le 4\sqrt{2}\varepsilon\sqrt{1+\varepsilon} \left(\int_0^{1+\varepsilon} (|g(x)|^2 + |g''(x)|^2)dx \right)^{\frac{1}{2}} \\
&\le 8\varepsilon.
\end{aligned}
$$

From this estimate and Lemma 3.2 we obtain (3.11).

The lower bound of the gap in (3.11) is attained by plugging in (3.8) the eigenfunction

$$
g_\varepsilon(x) = c_\varepsilon \sin \frac{\pi x}{1+\varepsilon}
$$

corresponding to the smallest nonzero eigenvalue $\left(\frac{\pi}{1+\varepsilon}\right)^2$, with the normalization constant $c_\varepsilon > 0$ such that

$$\|g_\varepsilon\|^2_{H^2(0,1+\varepsilon)} = \int_0^{1+\varepsilon} [g_\varepsilon^2(x) + g_\varepsilon''^2(x)]dx = 1,$$

and by applying the lower bound of k in Lemma 3.2. Through simple calculations we attain

$$c_\varepsilon = \left[\frac{1}{2}\left(1+\varepsilon+\frac{\pi^2}{1+\varepsilon}\right)\right]^{-\frac{1}{2}}.$$

Thus

$$\delta(S_\varepsilon, S) > \frac{1}{4}\left(1+\varepsilon+\frac{\pi^2}{1+\varepsilon}\right)^{-\frac{1}{2}} \sin\frac{\pi}{1+\varepsilon},$$

and from this, using

$$\sin\frac{\pi}{1+\varepsilon} > \varepsilon,$$

the get the first inequality in (3.11). □

Remark 3.1. *For the constant B that appears in Proposition 3.1 we have*

$$\frac{\pi}{4\sqrt{1+\pi^2}} \leq B \leq \frac{8}{\sqrt{3}}.$$

Another way that one can think of extending the definition of gap for operators defined on different open sets is to use extension by zero of the functions involved in the definition. Let us first agree, that if a function u defined on $\Omega \subset \mathbb{R}^n$ has been given, to denote with u_0 its extension to $\mathbb{R}^n$ by assigning to this extension the value 0 in $\mathbb{R}^n - \Omega$.

Let $\Omega_1 \subset \mathbb{R}^n$ and $\Omega_2 \subset \mathbb{R}^n$ be open sets and

$$S : \mathrm{D}(S) \subset L^2(\Omega_1) \to L^2(\Omega_1),$$

$$T : \mathrm{D}(T) \subset L^2(\Omega_2) \to L^2(\Omega_2)$$

linear operators. Then we can define the gap between these operators as follows:

$$\delta_\#(S,T) = \sup_{\substack{u \in \mathrm{D}(S) \\ \|u\|^2_{L^2(\Omega_1)} + \|Su\|^2_{L^2(\Omega_1)} = 1}} \inf_{v\in \mathrm{D}(T)} \left(\|u_0 - v_0\|^2_{L^2(\Omega_1\cup\Omega_2)} + \|(Su)_0 - (Tv)_0\|^2_{L^2(\Omega_1\cup\Omega_2)}\right)^{\frac{1}{2}} \tag{3.12}$$

and

$$\tilde{\delta_0}(S,T) = \max[\delta_0(S,T), \delta_0(T,S)]. \tag{3.13}$$

Clearly $\hat{\delta_0}(S,T) \leq \hat{\delta_{\#0}}(S,T)$ and $\hat{\delta_\#}(S,T) = \hat{\delta_0}(S,T)$ if $\Omega_1 = \Omega_2$.

However, this definition is not good, because if we apply it to the operators S, S_ε of Example 3.2, we find out that $\delta_\#(S_\varepsilon, S)$ is not infinitesimal as $\varepsilon \to 0^+$. Indeed, if $g \in \mathrm{D}(S_\varepsilon)$, using (3.8), we have

$$\inf_{f\in \mathrm{D}(S)} \int_0^{1+\varepsilon} [(g_0 - f_0)^2 + ((g'')_0 - (f'')_0)^2] dx \quad =$$

$$= \inf_{f\in \mathrm{D}(S)} \int_0^1 [(g-f)^2 + (g''-f'')^2]dx + \int_1^{1+\varepsilon} (g^2 + (g'')^2)dx =$$

$$= k^2 g(1)^2 + \int_1^{1+\varepsilon} (g^2 + (g'')^2)dx,$$

where the constant $k > 0$ is independent of g. Hence, our problem becomes to estimate

$$\delta_{\#}(S_\varepsilon, S)^2 \quad = \sup_{\substack{g \in C^2(I_\varepsilon)\cap C(\overline{I_\varepsilon}) \\ g(0)=g(1+\varepsilon)=0 \\ \int_0^{1+\varepsilon} (g^2+(g'')^2)dx = 1}} \left(k^2|g(1)|^2 + \int_1^{1+\varepsilon} [g^2 + (g'')^2]dx \right).$$

We choose a function $g_\varepsilon \in C_c^\infty(\mathbb{R})$ with the support contained in the interval $]1, 1+\varepsilon[$ and such that

$$\int_0^{1+\varepsilon} [g_\varepsilon^2 + (g_\varepsilon'')^2]dx = \int_1^{1+\varepsilon} [g_\varepsilon^2 + (g_\varepsilon'')^2]dx = 1.$$

Plugging g_ε into its expression, we obtain

$$\delta_{\#}(S_\varepsilon, S) \geq 1,$$

and this independently from ε. Thus, it becomse appearent that $\delta_{\#}$ can not serve as a useful notion of gap.

3.2 Main definition and examples

In the present subsection we propose the main definition of gap between operators defined on different open sets.

Let Γ be a family of open sets of $\mathbb{R}^n$ $(n \in \mathbb{N})$, and suppose that to each open set $\Omega \in \Gamma$ a linear operator

$$S_\Omega : \mathrm{D}(S_\Omega) \subset X(\Omega) \to Y(\Omega),$$

is associated, where $X(\Omega)$, $Y(\Omega)$ are normed spaces and the linear subspace $\mathrm{D}(S_\Omega)$ of $X(\Omega)$, is the domain of the operator S_Ω. Thus Γ serves to determine the class of operators that will be the object of the study of domain perturbation problems. In applications Γ will be as vast as possible and the operators S_Ω will be of the same nature, like say, second order elliptic operators with boundary conditions of the same type.

However, the perturbation of the domain will not be allowed within the entire family Γ, but only within a subfamily Λ of Γ. On the subfamily of open sets Λ we shall require that it satisfies some additional conditions: if $\Omega_1,\ \Omega_2 \in \Lambda$, then $\Omega_1 \setminus \overline{\Omega_2},\ \Omega_2 \setminus \overline{\Omega_1} \in \Gamma$. Moreover, we shall adapt the convention that $\emptyset \in \Gamma$; this has as a consequence that if $\Omega_1 \subset \Omega_2$ then $\Omega_1 \setminus \overline{\Omega_2}$ automatically belongs to Γ.

We shall also make use of the notions of direct sum space of two given linear spaces and of direct sum operator of two given linear operators as defined in §2.7. To the empty set $\emptyset$ will be associated, by convention, the null spaces $X(\emptyset) = Y(\emptyset) = \{0\}$, so that $X(\Omega)\oplus X(\emptyset) = X(\Omega)$, $Y(\Omega)\oplus Y(\emptyset) = Y(\Omega)$ and $S_\Omega \oplus S_\emptyset = S_\Omega$.

Finally, we shall assume that if $\Omega_1,\ \Omega_2 \in \Lambda$ and $\Omega_1 \cap \Omega_2 = \emptyset$ then $\Omega_1 \cup \Omega_2 \in \Gamma$ and

$$X(\Omega_1) \oplus X(\Omega_2) = X(\Omega_1 \cup \Omega_2) \quad \text{and} \quad Y(\Omega_1) \oplus Y(\Omega_2) = Y(\Omega_1 \cup \Omega_2).$$

The equality of the normed spaces here should be understood in the sense that between them (the normed spaces) there exists a norm-preserving linear bijective mapping.

The main example that we had in mind when introducing the previous notation is the following.

Example 3.3. *Let Γ be the set of all open subsets of $\mathbb{R}^n$ and, for each $\Omega \in \Gamma$, let $X(\Omega) = Y(\Omega) = L^2(\Omega)$, (actually, the present discourse holds for any $1 \le p \le \infty$), and let S_Ω be an arbitrary (partially defined) linear operator in $L^2(\Omega)$. Let also Λ be any subfamily of Γ (in the applications it is usually required that the boundaries of the elements of Λ satisfy certain regularity properties, ecc). It is easy to check that the previous requirements are satisfied by the families Γ, Λ, $(X(\Omega))_{\Omega\in\Gamma}$ and $(Y(\Omega))_{\Omega\in\Gamma}$.*

With these notations we give the following definition.

Definition 3.2. *Let $\Omega_1,\ \Omega_2 \in \Lambda$. We define the gap between the operators $S_{\Omega_1},\ S_{\Omega_2}$ by*

$$\delta(S_{\Omega_1},\ S_{\Omega_2}) = \delta(S_{\Omega_1} \oplus S_{\Omega_2\setminus\overline{\Omega_1}},\ S_{\Omega_2} \oplus S_{\Omega_1\overline{\Omega_1}}). \tag{3.14}$$

and

$$\hat{\delta}(S_{\Omega_1},\ S_{\Omega_2}) = \max\{\delta(S_{\Omega_1},\ S_{\Omega_2}),\ \delta(S_{\Omega_2},\ S_{\Omega_1})\}, \tag{3.15}$$

where the gaps n the right side members of the above equations are gaps between operators in the space $L^2(\Omega_1 \cup \Omega_2)$ according to Definition 1.3.

Remark 3.2. *If for each $\Omega \in \Lambda$, the normed space $X(\Omega)$ is a Hilbert space and the operator S_Ω is self-adjoint (or essentially self-adjoint), then, if $\Omega_1, \Omega_2 \in \Gamma$, since the operators $S_{\Omega_1} \oplus S_{\Omega_2\setminus\overline{\Omega_1}}$ are self-adjoint, the gap between S_{Ω_1} and S_{Ω_2} is symmetric;*

$$\delta(S_{\Omega_1},\ S_{\Omega_2}) = \delta(S_{\Omega_2},\ S_{\Omega_1}) = \hat{\delta}(S_{\Omega_2},\ S_{\Omega_1}),$$

see Proposition 2.1 (vi) and Corollary 1.1.

We take into consideration again the operators of Example 3.2 in the framework of the new notations and the new definition of gap introduced here, and estimate the gap between them.

Example 3.4. *Let Γ be the family of all open intervals of $\mathbb{R}$. For each interval $I =]a, b[$, $a, b \in \mathbb{R}$, is defined the operator*

$$S_I : \mathrm{D}(S_I) \subset L^2(I) \to L^2(I)$$

in the following way:

$$\mathrm{D}(S_I) = \{f \in C^2(I) \cap C(\overline{I}) : f'' \in L^2(I) \quad f(0) = f(1) = 0\}$$

and

$$S_I f = -f'' \quad \text{for each} f \in \mathrm{D}(S_I).$$

Let Λ be the set of all intervals of the kind $I_\varepsilon =]0, 1+\varepsilon[$ for $0 \leq \varepsilon \leq 1$. For each $0 \leq \varepsilon \leq 1$ the operator S_{I_ε} is denoted by S_ε and is the same operator as in Example 3.2. We set $S = S_0$. For brevity we define

$$S'_\varepsilon = S \oplus S_{I_\varepsilon \setminus \overline{I}}.$$

As we have already remarked $L^2(I_\varepsilon)$ is canonically identifiable with $L^2(I) \oplus L^2(1, 1+\varepsilon)$ (note that $I_\varepsilon \setminus \overline{I} =]1, 1+\varepsilon[$) and therefore the domain of the operator S'_ε can be seen as the set

$$\begin{aligned}\mathrm{D}(S'_\varepsilon) = \mathrm{D}(S) \oplus \mathrm{D}(S_{I_\varepsilon \setminus \overline{I}}) = \{& f \in C^2(I) \cap C^2(I_\varepsilon \setminus \overline{I}) \cap C(\overline{I_\varepsilon}) : \\ & f'' \in L^2(I) \; f'' \in L^2(I_\varepsilon \setminus I) \; f(0) = f(1) = f(1+\varepsilon) = 0\},\end{aligned}$$

and

$$S'_\varepsilon f = -f'' \in L^2(0, 1+\varepsilon)$$

for all $f \in \mathrm{D}(S'_\varepsilon)$. We have

$$\delta(S_\varepsilon, S) = \delta(S_\varepsilon, S'_\varepsilon) = \sup_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0 \\ \int_0^{1+\varepsilon} (g^2 + (g'')^2) dx = 1}} \mathrm{dist}_{H^2(0,1+\varepsilon)}(g, \; \mathrm{D}(S'_\varepsilon)), \tag{3.16}$$

where[17]

$$\mathrm{dist}_{H^2(0,1+\varepsilon)}(g, \; \mathrm{D}(S'_\varepsilon)) = \inf_{\substack{f \in C^2(I) \cap C^2(I_\varepsilon \setminus \overline{I}) \cap \\ C(\overline{I_\varepsilon}) \\ f(0) = f(1) = 0 \\ f(1+\varepsilon) = 0}} \left(\int_0^{1+\varepsilon} [(g-f)^2 + (g''-f'')^2] dx \right)^{\frac{1}{2}}.$$

With the same reasoning as in the proof of Lemma 3.1 we obtain

$$\mathrm{dist}_{H^2(0,1+\varepsilon)}(g, \; \mathrm{D}(S'_\varepsilon)) = k(\varepsilon)|g(1)|, \tag{3.17}$$

where

$$k(\varepsilon) = \inf_{\substack{v \in C^2(I) \cap C^2(I_\varepsilon \setminus \overline{I}) \cap C(\overline{I_\varepsilon}) \\ v(0) = v(1+\varepsilon) = 0 \\ v(1) = 1}} \left(\int_0^{1+\varepsilon} [v^2 + (v'')^2] dx \right)^{\frac{1}{2}}. \tag{3.18}$$

From Lemma 3.2, it is fairly easy to observe that $k(\varepsilon) \geq k \geq 1/(4\sqrt{2})$, where k is the constant that appears in that lemma. Un upper bound of $k(\varepsilon)$ can be obtained in the following way: let $u \in C^2(I) \cap C(\overline{I})$, we extend it by defining

[17] The reader should be aware that we are not claiming that $\mathrm{D}(S'_\varepsilon) \subset H^2(0, 1+\varepsilon)$, actually it does not. Nevertheless, the norm $\| \cdot \|_{H^2(0,1+\varepsilon)}$ can be evaluated on the elements of $\mathrm{D}(S'_\varepsilon)$ assuming finite values and, it is by the means of this norm that the distance of g from $\mathrm{D}(S'_\varepsilon)$ is being calculated.

$u(x) = (1+\varepsilon-x)/\varepsilon$ for $1 \le x \le 1+\varepsilon$. Plugging u in the right side integral of (3.18)*, and taking into account the arbitrariety of the initial u, we find that*

$$k \le k(\varepsilon) \le \sqrt{k^2 + \frac{\varepsilon}{3}}.$$

Thus

$$\lim_{\varepsilon \to 0^+} k(\varepsilon) = k$$

and since we are interested for small ε, ($0 \le \varepsilon \le 1$), an upper bound for $k(\varepsilon)$, by Lemma 3.2, is $\sqrt{2/3}$.

Plugging (3.17) *into* (3.16) *we obtain that*

$$\delta(S_\varepsilon, S) = k(\varepsilon) \cdot \sup_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0 \\ \int_0^{1+\varepsilon} (g^2 + (g'')^2) dx = 1}} |g(1)|.$$

Thus, it remains to estimate the supremum in the second side of the previous equality, but fortunately this job has already been done during the proof of Proposition 3.1. We find that

$$\delta(S_\varepsilon, S) = B\varepsilon + O(\varepsilon^2) \tag{3.19}$$

as $\varepsilon \to 0^+$, where the constant $B > 0$ is the same that appears in Proposition 3.1.

We can also give lower and upper bounds for the gap. From the above considerations and the proof of Lemma 3.3, we have

$$\frac{1}{4\sqrt{2+\pi^2}}\varepsilon \le \delta_0(S_\varepsilon, S) \le \delta(S_\varepsilon, S) \le \frac{8\sqrt{2}}{\sqrt{3}}\varepsilon \tag{3.20}$$

for $0 < \varepsilon \le 1$.

3.2.1 Gap between operators of boundary-value problems on different open sets

As we said the definition of gap between operators defined on different open sets that we shall adapt is Definition 3.2, which was given in quite general terms.

A more specific situation is the following which takes into account operators that derive from boundary-value problems for partial differential equations.

Let $\Omega_0 \subset \mathbb{R}^n$ be an open set and let

$$L : \mathrm{D}(L) \subset L^2_{\mathrm{loc}}(\Omega_0) \to L^2_{\mathrm{loc}}(\Omega_0)$$

be a linear operator. Furthermore, we will assume that the operator L is *local*, i.e., for any open set $\Omega \subset \Omega_0$ and $U, V \in \mathrm{D}(L)$ such that $U|_\Omega = V|_\Omega$, the equality $(LU)|_\Omega = (LV)|_\Omega$ holds (for instance L might be the operator that originates from some differential expression). This permits us to consider for each bounded open set Ω such that $\overline{\Omega} \subset \Omega_0$ the following operator

$$L|_\Omega : \mathrm{D}(L_\Omega) \subset L^2(\Omega) \to L^2(\Omega),$$

with

$$\mathrm{D}(L|_\Omega) = \mathrm{D}(L)|_\Omega = \{U|_\Omega : U \in \mathrm{D}(L)\}$$

and for all $u \in \mathrm{D}(L|_\Omega)$

$$L|_\Omega u = (LU)|_\Omega,$$

where $U \in \mathrm{D}(L)$ is such that $U|_\Omega = u$. Let also be given a family Γ of open bounded subsets Ω of Ω_0 such that $\overline{\Omega} \subset \Omega_0$ and that might satisfy certain additional properties that can depend from the nature of the operator L considered, the boundary value operator or simply by the fact that one might want to study only a certain class of problems. Assume that for each $\Omega \in \Gamma$ a boundary operator

$$B_\Omega : \mathrm{D}(B_\Omega) \subset L^2(\Omega) \to \left(L^2(\partial\Omega)\right)^m,$$

where $m \in \mathbb{N}$, is defined satisfying $\mathrm{D}(L)|_\Omega \subset \mathrm{D}(B_\Omega)$.

For each $\Omega \in \Gamma$ consider the operator

$$S_\Omega : \mathrm{D}(S_\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

defined by

$$\mathrm{D}(S_\Omega) = \{u \in \mathrm{D}(L)|_\Omega : B_\Omega u = 0\}$$

and for all $u \in \mathrm{D}(S_\Omega)$

$$S_\Omega u = L|_\Omega u = (LU)|_\Omega,$$

where $U \in \mathrm{D}(L)$ is such that $U|_\Omega = u$.

Let $\Omega_1, \Omega_2 \in \Gamma$; in particular they are bounded open sets such that $\overline{\Omega_1}, \overline{\Omega_2} \subset \Omega_0$. Let the operator

$$S_{\Omega_1,\,\Omega_2} : \mathrm{D}(S_{\Omega_1,\,\Omega_2}) \subset L^2(\Omega_1 \cup \Omega_2) \to L^2(\Omega_1 \cup \Omega_2)$$

be defined by

$$D\left(S_{\Omega_1,\,\Omega_2}\right) = \{u \in \mathrm{D}(L)|_{\Omega_1 \cup \Omega_2} : u|_{\Omega_1} \in \mathrm{D}(S_{\Omega_1}) \quad \textit{and} \quad u|_{\Omega_2 \setminus \overline{\Omega_1}} \in \mathrm{D}(S_{\Omega_2 \setminus \overline{\Omega_1}})\}$$

and

$$S_{\Omega_1,\,\Omega_2} u = \begin{cases} S_{\Omega_1}(u|_{\Omega_1}) & = \quad L|_{\Omega_1}(u|_{\Omega_1}) \qquad\quad \textit{in } \Omega_1 \\ S_{\Omega_2 \setminus \overline{O_1}}(u|_{\Omega_2 \setminus \overline{O_1}}) & = \quad L|_{\Omega_2 \setminus \overline{O_1}}(u|_{\Omega_2 \setminus \overline{\Omega_1}}) \quad \textit{in } \Omega_2 \setminus \overline{\Omega_1} \end{cases}$$

for all $u \in \mathrm{D}(S_{\Omega_1,\,\Omega_2})$.

Analogously is defined $S_{\Omega_2,\,\Omega_1}$, just by reversing the roles of Ω_2 and Ω_1. We will be particularly interested to the case when $\Omega_1 \subset \Omega_2$. In this situation it is $S_{\Omega_2,\,\Omega_1} = S_{\Omega_2}$.

Remark 3.3. *Let $\Omega_1, \Omega_2 \in \Gamma$. Then we have the canonical identifications:*

$$L^2(\Omega_1 \cup \Omega_2) = L^2(\Omega_1) \oplus L^2(\Omega_2 \setminus \Omega_1),$$

$$S_{\Omega_1,\Omega_2} = S_{\Omega_1} \oplus S_{\Omega_2 \setminus \overline{\Omega_1}},$$

therefore

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \delta\left(S_{\Omega_1,\,\Omega_2}, S_{\Omega_2,\,\Omega_1}\right) \tag{3.21}$$

and

$$\hat{\delta}(S_{\Omega_1}, S_{\Omega_2}) = \max[\delta(S_{\Omega_1}, S_{\Omega_2}), \delta(S_{\Omega_2}, S_{\Omega_1})], \tag{3.22}$$

where the gaps on the right side members of the above equations are being taken in the space $L^2(\Omega_1 \cup \Omega_2)$.

We might say that the definition exposed above is the natural way of introducing linear operators related to boundary value problems associated with linear partial differential equations. In the case of certain elliptic equations one can make also use of non-negative quadratic forms, as in §2.2, in particular Theorem 2.2, to introduce non-negative self-adjoint operators that extend classical linear operators related to boundary value problems associated with these equations (as an example see the case of the Dirichlet Laplacian in §2.6). And of course it makes sense, and actually, it is important to evaluate the gap between these kind of linear operators derived from quadratic forms.

Example 3.5. *Let Γ be the class of open bounded subsets of $\mathbb{R}^n$ with a $C^{1,r}$ boundary for some $\frac{1}{2} < r < 1$. Let $\Omega_1, \Omega_2 \in \Gamma$. We consider the generalized minus Dirichlet Laplacian operators $-\Delta_{\Omega_i,\mathrm{D}}$ associated to the open sets Ω_i $(i = 1, 2)$ respectively as outlined in Theorem 2.8. For the gap between these two operators we have*

$$\delta(-\Delta_{\Omega_1,\mathrm{D}},\, -\Delta_{\Omega_1,\mathrm{D}}) = \sup_{\substack{u \in H_0^1(\Omega_1) \cap H^\Delta(\Omega_1) \\ \cap H_0^1(\Omega_2 \setminus \overline{\Omega_1}) \cap H^\Delta(\Omega_2 \setminus \overline{\Omega_1}) \\ \|u\|^2_{H^\Delta(\Omega_1)} + \|u\|^2_{H^\Delta(\Omega_2 \setminus \overline{\Omega_1})} = 1}} \quad \inf_{\substack{v \in H_0^1(\Omega_2) \cap H^\Delta(\Omega_2) \\ \cap H_0^1(\Omega_1 \setminus \overline{\Omega_2}) \cap H^\Delta(\Omega_1 \setminus \overline{\Omega_2})}} \mathcal{H}(u, v),$$

where

$$\begin{aligned}\mathcal{H}(u,v) \;=\; & \Big(\|u - v\|^2_{L^2(\Omega_1 \cup \Omega_2)} + \|\Delta u - \Delta v\|^2_{L^2(\Omega_1 \cap \Omega_2)} \\ + \;& \|\Delta u - \Delta v\|^2_{L^2(\Omega_2 \setminus \overline{\Omega_1})} + \|\Delta u - \Delta v\|^2_{L^2(\Omega_1 \setminus \overline{\Omega_2})}\Big)^{\frac{1}{2}}\end{aligned}$$

for all $u \in H^\Delta(\Omega_1) \cap H^\Delta(\Omega_2 \setminus \overline{\Omega_1})$ and $v \in H^\Delta(\Omega_2) \cap H^\Delta(\Omega_1 \setminus \overline{\Omega_2})$.

Actually, in the above formula the regularity assumption on the boundaries of Ω_i, $i = 1, 2$, is not needed. But under these hypotheses the open sets Ω_i, $i = 1, 2$, satisfy the regularity assumptions of Theorem 2.9, and therefore in the previous expression the space $H^\Delta(\Omega_i)$ can be replaced with the Sobolev space $H^2(\Omega_i)$ $(i = 1, 2)$. If moreover, the dimension of the space $n \leq 3$ so that the approximation Theorem 2.10 holds, then

$$\delta(-\Delta_{\Omega_1,\mathrm{D}},\, -\Delta_{\Omega_1,\mathrm{D}}) = \sup_{\substack{u \in C(\overline{\Omega_1 \cup \Omega_2}) \cap C^2(\Omega_1) \\ \cap H^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \\ \cap H^2(\Omega_2 \setminus \overline{\Omega_1}) \\ u|_{\partial\Omega_1} = 0,\ u|_{\partial(\Omega_2 \setminus \overline{\Omega_1})} = 0 \\ \|u\|^2_{H^\Delta(\Omega_1)} + \|u\|^2_{H^\Delta(\Omega_2 \setminus \overline{\Omega_1})} = 1}} \quad \inf_{\substack{v \in C(\overline{\Omega_1 \cup \Omega_2}) \cap C^2(\Omega_2) \\ \cap H^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2}) \\ \cap H^2(\Omega_1 \setminus \overline{\Omega_2}) \\ v|_{\partial\Omega_2} = 0,\ v|_{\partial(\Omega_1 \setminus \overline{\Omega_2})} = 0}} \mathcal{H}(u, v)$$

The other gap $\delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}})$ is obtained by interchanging Ω_1 and Ω_2. Actually these gaps are equal because the operators $-\Delta_{\Omega,\mathrm{D}}$ are self-adjoint for all bounded open sets Ω, and this implies that both the operators $S_{\Omega_1,\Omega_2} = S_{\Omega_1} \oplus S_{\Omega_2 \setminus \overline{\Omega_1}}$, $S_{\Omega_2,\Omega_1} = S_{\Omega_2} \oplus S_{\Omega_1 \setminus \overline{\Omega_2}}$ are self-adjoint (see the previous Remark 3.3, Proposition 2.1 and Corollary 1.1).

If $\Omega_1 \subset \Omega_2$ this expression simplifies somehow as we shall see in §4.1.

3.3 Another possible definition for the gap between operators defined on different open sets

Next we expose a new definition of gap.

Let $\Omega_0 \subset \mathbb{R}^n$ be an open set and let

$$L : \mathrm{D}(L) \subset L^2_{\mathrm{loc}}(\Omega_0) \to L^2_{\mathrm{loc}}(\Omega_0)$$

be a linear operator. Furthermore, we will assume that the operator L is *local*, i.e., for any open set $\Omega \subset \Omega_0$ and $U, V \in \mathrm{D}(L)$ such that $U|_\Omega = V|_\Omega$, the equality $(LU)|_\Omega = (LV)|_\Omega$ holds. This permits us to consider for each such an Ω the following operator

$$L|_\Omega : \mathrm{D}(L_\Omega) \subset L^2(\Omega) \to L^2(\Omega),$$

with

$$\mathrm{D}(L|_\Omega) = \mathrm{D}(L)|_\Omega = \{U|_\Omega : U \in \mathrm{D}(L)\}$$

and for all $u \in \mathrm{D}(L|_\Omega)$

$$L|_\Omega u = (LU)|_\Omega,$$

where $U \in \mathrm{D}(L)$ is such that $U|_\Omega = u$. Let also be given a family Γ of open bounded subsets Ω of Ω_0 such that $\overline{\Omega} \subset \Omega_0$ and that satisfy certain additional properties to be specified later. Assume that for each $\Omega \in \Gamma$ a boundary operator

$$B_\Omega : \mathrm{D}(B_\Omega) \subset L^2(\Omega) \to \left(L^2(\partial\Omega)\right)^m,$$

where $m \in \mathbb{N}$, is defined satisfying $\mathrm{D}(L)|_\Omega \subset \mathrm{D}(B_\Omega)$.

For each $\Omega \in \Gamma$ consider the operator

$$S_\Omega : \mathrm{D}(S_\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

defined by

$$\mathrm{D}(S_\Omega) = \{u \in \mathrm{D}(L)|_\Omega : B_\Omega u = 0\}$$

and for all $u \in \mathrm{D}(S_\Omega)$

$$S_\Omega u = L|_\Omega u = (LU)|_\Omega,$$

where $U \in \mathrm{D}(L)$ is such that $U|_\Omega = u$.

Let $\Omega_1, \Omega_2 \in \Gamma$; in particular they are bounded open sets such that $\overline{\Omega}_1, \overline{\Omega}_2 \subset \Omega_0$. Let the operator S_{Ω_1, Ω_2} be defined by

$$D\left(S_{\Omega_1, \Omega_2}\right) = \{u \in \mathrm{D}(L)|_{\Omega_1 \cup \Omega_2} : B_{\Omega_1}(u|_{\Omega_1}) = 0\}$$

and

$$S_{\Omega_1, \Omega_2} u = L|_{\Omega_1 \cup \Omega_2} u \quad \forall u \in D\left(S_{\Omega_1, \Omega_2}\right).$$

Definition 3.3. *Let $\Omega_1, \Omega_2 \in \Gamma$. Then we define*

$$\delta_*(S_{\Omega_1}, S_{\Omega_2}) = \delta\left(S_{\Omega_1, \Omega_2}, S_{\Omega_2, \Omega_1}\right) \tag{3.23}$$

and

$$\hat{\delta}_*(S_{\Omega_1}, S_{\Omega_2}) = \max[\delta_*(S_{\Omega_1}, S_{\Omega_2}), \delta_*(S_{\Omega_2}, S_{\Omega_1})], \tag{3.24}$$

where the gaps on the right side members of the above equations are being taken in the space $L^2(\Omega_1 \cup \Omega_2)$.

Example 3.6. *Let Γ be the class of open bounded sets in $\mathbb{R}^n$, $\mathrm{D}(L) = C^2(\mathbb{R}^n)$, $Lu = -\Delta u$ for all $u \in C^2(\mathbb{R}^n)$ and for a bounded open set $\Omega \in \Gamma$ let $\mathrm{D}(B_\Omega) = C(\overline{\Omega})$ and $B_\Omega u = u|_{\partial\Omega}$ for all $u \in C(\overline{\Omega})$. Hence*

$$\mathrm{D}(S_\Omega) = \{u \in C^2(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$$

and

$$S_\Omega u = -\Delta u \quad \text{for all } u \in C(\overline{\Omega}).$$

Let Ω_1, Ω_2 be bounded open sets in $\mathbb{R}^n$. Then, in the notations of Example 3.1, $S_{\Omega_1} = S$ and $S_{\Omega_2} = T$ and, if we denote for brevity $\Omega = \Omega_1 \cup \Omega_2$,

$$\delta_*(S,T) = \sup_{\substack{u \in C^2(\overline{\Omega}),\ u|_{\partial\Omega_1} = 0 \\ \|u\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} = 1}} \ \inf_{\substack{v \in C^2(\overline{\Omega}) \\ v|_{\partial\Omega_2} = 0}} \left(\|u - v\|^2_{L^2(\Omega)} + \|\Delta u - \Delta v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Example 3.7. *In Example 3.6 let $n = 1$, $\Omega_1 =]0,1[:= I$, $\Omega_2 =]0, 1+\varepsilon[:= I_\varepsilon$ for any $\varepsilon > 0$. In this case in the notation of Example 3.2 $S_{\Omega_1} = S$, $S_{\Omega_2} = S_\varepsilon$ and*

$$\delta_*(S, S_\varepsilon) = \sup_{\substack{f \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ f(0) = f(1) = 0 \\ \int_0^{1+\varepsilon} (f^2 + (f'')^2) dx = 1}} \ \inf_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0}} \left(\int_0^{1+\varepsilon} \left((f-g)^2 + (f''-g'')^2\right) dx\right)^{\frac{1}{2}},$$

$$\delta_*(S_\varepsilon, S) = \sup_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0 \\ \int_0^{1+\varepsilon} (g^2 + (g'')^2) dx = 1}} \ \inf_{\substack{f \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ f(0) = f(1) = 0}} \left(\int_0^{1+\varepsilon} \left((g-f)^2 + (g''-f'')^2\right) dx\right)^{\frac{1}{2}}.$$

For simplicity we introduce also the notations $S_{0,\varepsilon} = S_{\Omega_1,\Omega_2}$ and $S_{\varepsilon,0}$.

It remains to estimate $\delta_*(S, S_\varepsilon)$ and $\delta_*(S_\varepsilon, S)$ as $\varepsilon \to 0^+$. For this purpose we need the following Lemma:

Lemma 3.4. *Let $\varepsilon > 0$.*

(i) Given $f \in C^2(0, 1+\varepsilon) \cap C([0, 1+\varepsilon])$ with $f(0) = 0$, then

$$\begin{aligned} \operatorname{dist}((f, -f''), G(S_{\varepsilon,0})) \ &:= \inf_{\substack{g \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ g(0) = g(1+\varepsilon) = 0}} \left(\int_0^{1+\varepsilon} \left((f-g)^2 + (f''-g'')^2\right) dx\right)^{\frac{1}{2}} \\ &= l(\varepsilon)|f(1+\varepsilon)|, \end{aligned} \tag{3.25}$$

where

$$l(\varepsilon) = \inf_{\substack{v \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ v(0) = 0\ v(1+\varepsilon) = 1}} \left(\int_0^{1+\varepsilon} \left(v^2 + (v'')^2\right) dx\right)^{\frac{1}{2}}. \tag{3.26}$$

(ii) Given $g \in C^2(0, 1+\varepsilon) \cap C([0, 1+\varepsilon])$ *with* $g(0) = 0$*, then*

$$\begin{aligned}\operatorname{dist}((g, -g''), G(S_{0,\varepsilon})) &:= \inf_{\substack{f \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ f(0)=f(1)=0}} \left(\int_0^{1+\varepsilon} \left((g-f)^2 + (g''-f'')^2 \right) dx \right)^{\frac{1}{2}} \\ &= k(\varepsilon)|g(1)|, \end{aligned} \tag{3.27}$$

where

$$k(\varepsilon) = \inf_{\substack{v \in C^2(I_\varepsilon) \cap C(\overline{I_\varepsilon}) \\ v(0)=0 \; v(1)=1}} \left(\int_0^{1+\varepsilon} \left(v^2 + (v'')^2 \right) dx \right)^{\frac{1}{2}}. \tag{3.28}$$

(iii) For $l(\varepsilon)$ *and* $k(\varepsilon)$*, given by* (3.26) *and* (3.28) *respectively, the following bounds hold:*

$$\frac{1}{4\sqrt{2}(1+\varepsilon)^{\frac{3}{2}}} \leq l(\varepsilon) \leq \frac{1}{\sqrt{3}}(1+\varepsilon)^{\frac{1}{2}}, \tag{3.29}$$

$$\frac{1}{4\sqrt{2}(1+\varepsilon)^{\frac{3}{2}}} \leq k(\varepsilon) \leq \frac{1}{\sqrt{3}}(1+\varepsilon)^{\frac{3}{2}}. \tag{3.30}$$

Proof. The proof of (3.25) and (3.27) is carried in a similar fashion as the first part of the proof of Lemma 3.2. Yet, the lower bounds in the point (iii) are obtained arguing in a similar manner as in Lemma 3.2, where the main result to be used is again inequality (3.26) of [9], page 88. The upper bounds in (3.29) and in (3.30) are obtained by plugging $v(x) = \frac{1}{1+\varepsilon}x$ and $v(x) = x$ in (3.26) and in (3.28) respectively. □

Lemma 3.5. *For all* $0 < \varepsilon < \frac{1}{2}$*;*

$$\frac{1}{16} \frac{\pi}{\sqrt{1+\pi^2}} \cdot \sqrt{\frac{2}{3}} \cdot \varepsilon < \delta_*(S, S_\varepsilon) < \frac{8\sqrt{2}}{\sqrt{3}}\varepsilon \tag{3.31}$$

and

$$\frac{1}{2}\left(\frac{3}{2} + \pi^2\right)^{-\frac{1}{2}} \cdot \varepsilon < \delta_*(S_\varepsilon, S) < \frac{16\sqrt{2}}{\sqrt{3}}\varepsilon. \tag{3.32}$$

Proof. The proof of these estimates is obtained by following the same steps of the proof of Lemma 3.3 and by using the previous Lemma 3.5 □

4 Estimates of the gap between operators defined on different open sets

4.1 The case of the Dirichlet Laplacian

Example 4.1. *Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ be bounded open sets with a $C^{1,r}$ boundary for some $\frac{1}{2} < r < 1$. In the notations of Example 3.5 $\Omega_i \in \Gamma$, $i = 1, 2$. As in Example 3.5 we consider the generalized Dirichlet Laplacian operators $-\Delta_{\Omega_1,\mathrm{D}}$, $-\Delta_{\Omega_2,\mathrm{D}}$ associated with the open sets Ω_1, Ω_2 respectively. Then, concerning the gap between $-\Delta_{\Omega_1,\mathrm{D}}$ and $-\Delta_{\Omega_2,\mathrm{D}}$ we have:*

$$\delta(-\Delta_{\Omega_1,\mathrm{D}}, -\Delta_{\Omega_2,\mathrm{D}}) = \sup_{\substack{u \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \\ \cap C(\overline{\Omega_2}) \\ u|_{\partial\Omega_1} = 0, \ u|_{\partial\Omega_2} = 0 \\ \mathcal{K}(u) = 1}} \ \inf_{\substack{v \in C^2(\Omega_2) \cap C(\overline{\Omega_2}) \\ \Delta v \in L^2(\Omega_2) \\ v|_{\partial\Omega_2} = 0,}} \mathcal{K}(u - v)$$

and

$$\delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}}) = \sup_{\substack{v \in C^2(\Omega_2) \cap C(\overline{\Omega_2}) \\ v|_{\partial\Omega_2} = 0 \\ \mathcal{K}(v) = 1}} \ \inf_{\substack{u \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \\ \cap C(\overline{\Omega_2}) \\ \Delta u \in L^2(\Omega_1) \cap L^2(\Omega_2 \setminus \overline{\Omega_1}) \\ u|_{\partial\Omega_1} = 0, \ u|_{\partial\Omega_2} = 0}} \mathcal{K}(v - u),$$

where

$$\mathcal{K}(w) := \left(\|w\|^2_{L^2(\Omega_2)} + \|\Delta w\|^2_{L^2(\Omega_1)} + \|\Delta w\|^2_{L^2(\Omega_2 \setminus \overline{\Omega_1})} \right)^{\frac{1}{2}}$$

for any $w \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \cap C(\overline{\Omega_2})$.

To estimate $\delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}})$ we need to prove the following Lemma 4.1. For this purpose it is necessary to know under which conditions the classical Dirichlet problem associated to Laplace equation with continuous boundary data is solvable. We recall the following fact: the Dirichlet problem associated to Laplace equation has a solution in $C^2(\Omega) \cap C(\overline{\Omega})$ for arbitrary continuous boundary values if and only if the boundary points are all *regular*. For this fact and the definition of *regular boundary points* of a domain see [31], pages 25-26 and, in particular, Theorem 3.14. For a domain to have regular boundary points sufficient general conditions can be given in terms of local geometrical properties (see again [31]). We limit ourselves to a simple one of such conditions - *the exterior sphere* condition: an open domain $\Omega \subset \mathbb{R}^n$ is said to satisfy *the exterior sphere condition at* $x_0 \in \partial\Omega$ if there exists an open ball $B_r(y) \subset \mathbb{R}^n - \overline{\Omega}$ such that $\overline{B_r(y)} \cap \Omega = \{x_0\}$. An open set Ω is said to satisfy *the exterior sphere condition* if it satisfies the exterior sphere condition at every point $x_0 \in \partial\Omega$. If in the above definition the value r of the radius of the ball $B_r(y)$ can be chosen independently of $x_0 \in \partial\Omega$ it said that Ω satisfies *the uniform exterior sphere condition.* In particular, it can be shown that a domain with a $C^{1,1}$ boundary satisfies the exterior sphere condition.

Furthermore, an open set Ω is said to satisfy *the interior sphere condition* at $x_0 \in \partial\Omega$ if there exists an open ball $B_r(y) \subset \Omega$ such that $\overline{B_r(y)} \cap (\mathbb{R}^n \setminus \Omega) = \{x_0\}$ (that is, the complement of Ω satisfies the exterior sphere condition at x_0). Analogously, an open set Ω is said to satisfy *the interior sphere condition* if Ω satisfies the interior sphere condition at any point $x_0 \in \partial\Omega$. Finally, an open set Ω is said to satisfy *the uniform interior sphere condition* if there exists $r > 0$ such that for every $x_0 \in \partial\Omega$ there exists a ball $B_r(y) \subset \Omega$ such that $\overline{B_r(y)} \cap (\mathbb{R}^n \setminus \Omega) = \{x_0\}$.

Lemma 4.1. *Let $\Omega_1 \subset \Omega_2$ be open bounded sets in $\mathbb{R}^n$ such that $\Omega_1, \Omega_2 \in C^{1,r}$ for some $\frac{1}{2} < r < 1$, Ω_1 satisfies both the interior and exterior sphere condition and, Ω_2 satisfies the exterior sphere condition.*[18] *Assume that $v \in C^2(\Omega_2) \cap C(\overline{\Omega_2})$, $\Delta v \in L^2(\Omega_2)$ and $v|_{\partial\Omega_2} = 0$*

Then

$$\begin{aligned} I(v) &:= \inf_{\substack{u \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \cap C(\overline{\Omega_2}) \\ u|_{\partial\Omega_1} = 0,\ u|_{\partial\Omega_2} = 0}} \mathcal{K}(v-u) \\ &\leq (\textit{meas } \Omega_2)^{\frac{1}{2}} \|v\|_{C(\partial\Omega_1)}. \end{aligned}$$

Proof. Note that

$$I(v) = \inf_{\substack{w \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \cap C(\overline{\Omega_2}) \\ w|_{\partial\Omega_1} = v|_{\partial\Omega_1},\ w|_{\partial\Omega_2} = 0}} \mathcal{K}(w). \tag{1}$$

Let $w' \in C^2(\Omega_1) \cap C(\overline{\Omega_1})$ be such that

$$\begin{cases} \Delta w' = 0 & \text{in } \Omega_1 \\ w'|_{\partial\Omega_1} = v|_{\partial\Omega_1} \end{cases}$$

and $w'' \in C^2(\Omega_2 \setminus \overline{\Omega_1}) \cap C(\overline{\Omega_2} \setminus \Omega_1)$ be such that

$$\begin{cases} \Delta w'' = 0 & \text{in } \Omega_2 \setminus \overline{\Omega_1} \\ w''|_{\partial\Omega_1} = v|_{\partial\Omega_1} \\ w''|_{\partial\Omega_2} = 0 \end{cases}.$$

Such functions w' and w'' exist due to the considerations made above about open sets satisfying the sphere condition as in our assumptions.

We put

$$w_0 = \begin{cases} w' & \text{in } \Omega_1 \\ w'' & \text{in } \overline{\Omega_2} \setminus \Omega_1 \end{cases}. \tag{2}$$

Therefore

$$\begin{aligned} I(v) &\leq \mathcal{K}(w_0) \\ &= \left(\int_{\Omega_2} w_0^2 dx\right)^{\frac{1}{2}} \\ &= \left(\int_{\Omega_1} w_0^2 dx + \int_{\Omega_2 \setminus \Omega_1} w_0^2 dx\right)^{\frac{1}{2}} \end{aligned}$$

By the maximum principle (see, for instance, [64], page 33, Theorem 6.2.)

$$\begin{aligned} I(v) &\leq (\text{meas } \Omega_1 \|w_0\|^2_{C(\overline{\Omega_1})} + \text{meas } \Omega_2 \setminus \Omega_1 \|w_0\|^2_{C(\overline{\Omega_2} \setminus \Omega_1)})^{\frac{1}{2}} \\ &= (\text{meas } \Omega_1 \|v\|^2_{C(\partial\Omega_1)} + \text{meas } \Omega_2 \setminus \Omega_1 \|v\|^2_{C(\partial\Omega_1)})^{\frac{1}{2}} \\ &= (\text{meas } \Omega_2)^{\frac{1}{2}} \|v\|_{C(\partial\Omega_1)}. \quad \square \end{aligned}$$

[18] This is equivalent to say that both Ω_1 and $\Omega_2 \setminus \overline{\Omega_1}$ satisfy the exterior sphere condition.

The following corollary is immediate.

Corollary 4.1. *Let $\Omega_1 \subset \Omega_2$ be open bounded sets in $\mathbb{R}^n$ such that $\Omega_1, \Omega_2 \in C^{1,r}$ for some $\frac{1}{2} < r < 1$ and both Ω_1 and $\Omega_2 \setminus \overline{\Omega_1}$ satisfy the exterior sphere condition. Then*

$$\delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}}) \leq (meas\ \Omega_2)^{\frac{1}{2}} \sup_{\substack{v \in C^2(\Omega_1) \cap C^2(\Omega_2 \setminus \overline{\Omega_1}) \\ \cap C(\overline{\Omega_2}) \\ v|_{\partial\Omega_2} = 0 \\ \|v\|^2_{L^2(\Omega_2)} + \|\Delta v\|^2_{L^2(\Omega_2)} = 1}} \|v\|_{C(\partial\Omega_1)} \tag{4.1}$$

To proceed further with the estimation of the gap we need to investigate under what conditions do we have the validity of estimates of the type

$$\sup_{x\in\Omega} (\mathrm{dist}(x, \partial\Omega))^{-\gamma} |v(x)| \leq M \left(\int_\Omega \left(|v|^2 + |\Delta v|^2\right) dx \right)^{1/2}, \tag{4.2}$$

where Ω is a bounded open set, $v \in C^2(\Omega) \cap C(\overline{\Omega})$, $v = 0$ on $\partial\Omega$ and $0 < \gamma \leq 1$, $M > 0$ are fixed constants independent of the function v. With this regard we have the following result.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set such that satisfies the exterior sphere condition and has the cone property. Assume also that the dimension $n \leq 3$. Then estimate* (4.2) *holds for any $v \in C^2(\Omega) \cap C(\overline{\Omega})$, where $\gamma = 1/2$ if $n = 3$, γ can be chosen to be any number in $(0, 1)$ if $n = 2$, and $\gamma = 1$ if $n = 1$; and $M > 0$ is independent of v.*

Proof. *Step 1.* Since Ω satisfies the exterior sphere condition, then by Lemma 8.1, Ch.3 in the book by O.A. Ladyzhenskaya and N.N. Uraltseva [38] on elliptic equations, 1973, we have

$$\begin{aligned} ||v||_{H^2(\Omega)} &:= \left(\int_\Omega \left(|v|^2 + |Dv|^2 + |D^2 v|^2\right) dx \right)^{1/2} \\ &\leq M_1 \left(\int_\Omega \left(|v|^2 + |\Delta v|^2\right) dx \right)^{1/2}. \end{aligned}$$

Step 2. By Sobolev imbedding theorem, which is true for domains with the cone property, v belongs to a Hölder space $C^\gamma(\Omega)$ for some $\gamma \in (0, 1]$ chosen as in our assumptions, but only if the dimension $n \leq 3$. In this case,

$$\sup_{x\in\Omega} (\mathrm{dist}(x, \partial\Omega))^{-\gamma} |v(x)| \leq \sup_{x,y\in\overline{\Omega}} \frac{|v(x) - v(y)|}{|x - y|^\gamma} \leq M_2 ||v||_{H^2(\Omega)},$$

which imply the desired estimate with $M = M_1 M_2$. □

Remark 4.1. *This argument may look too rough, however, we cannot extend* (4.2) *to $n = 4$. Indeed, consider the function*

$$u(x) = (-\ln|x|)^\alpha - (\ln 2)^\alpha \qquad in \quad \Omega = \left\{ x \in \mathbb{R}^4 : \ |x| < \frac{1}{2} \right\},$$

where $0 < \alpha < \frac{1}{2}$. *Then*

$$|\Delta u| \leq const \cdot (-\ln |x|)^{\alpha-1}|x|^{-2} \in L_2(\Omega),$$

so that the right side in (4.2) *is finite, while the left side is* $+\infty$.

Finally, we have:

Theorem 4.1. *Let* $n \leq 3$, $\gamma = 1/2$ *if* $n = 3$, $0 < \gamma < 1$ *if* $n = 2$ *and* $\gamma = 1$ *if* $n = 1$. *Let* Ω_2 *be an open bounded set in* $\mathbb{R}^n$ *with a* $C^{1,r}$ *boundary for some* $\frac{1}{2} < r < 1$ *and that satisfies the exterior sphere condition. Then there exist* $M > 0$ *such that*

$$\delta(-\Delta_{\Omega_2,\mathrm{D}}, -\Delta_{\Omega_1,\mathrm{D}}) \leq M\varepsilon^{\gamma}, \tag{4.3}$$

for all $\varepsilon > 0$ *and for all open sets* $\Omega_1 \in C^{1,r}$ $(\frac{1}{2} < r < 1)$ *for which* $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^{\varepsilon}$ *and* $\Omega_2 \setminus \overline{\Omega_1}$ *satisfies the exterior sphere condition.*

Proof. This is a direct consequence of Corollary 4.1 and Lemma 4.2. Indeed, let $v \in C^2(\Omega_2) \cap C(\overline{\Omega_2})$ be such that $\mathcal{K}(v) = 1$, $v|_{\partial\Omega_2} = 0$ and $x \in \partial\Omega_1$. Let also $y \in \partial\Omega_2$ be such that $\mathrm{dist}(x, \partial\Omega_2) = |x - y|$. Then, if $x \notin \partial\Omega_2$, we have

$$\begin{aligned}
|v(x)| &= |x-y|^{\gamma}\frac{|v(x)|}{|x-y|^{\gamma}} \\
&\leq \varepsilon^{\gamma} \sup_{x\in\Omega_2} \frac{|v(x)|}{\mathrm{dist}(x,\partial\Omega_2)^{\gamma}} \\
&\leq M_0\varepsilon^{\gamma}\mathcal{K}(v) \\
&= M_0\varepsilon^{\gamma},
\end{aligned}$$

where $M_0, \gamma > 0$ are the constants that appear when applying Lemma 4.2 to the domain Ω_2. This estimate holds even if $x \in \partial\Omega_2$, for in this case simply $v(x) = 0$. Hence, by (4.1), we obtain (4.3) with $M = M_0\,(\mathrm{meas}\;\Omega_2)^{\frac{1}{2}}$. □

4.2 Extension of gap estimates to general elliptic operators

The estimates of §4.1 about the gap between the Dirichlet Laplacian operators defined on different open sets extend straightforwardly to more generic elliptic operators. Actually, in attaining the aforementioned results, the only ingredients that we have used were: the solvability of the corresponding homogeneous equation (Laplace equation) for continuous boundary data, the maximum principle, an H^2 global estimate and a Sobolev imbedding theorem. Well, these facts hold for a vast class of elliptic operators. For the sake of clarity, in the sequel we shall summarily repeat the procedure.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain. Let $(a_{ij})_{i,j=1,\ldots,n}$ be a Hermitian matrix of Lipschitz continuous functions and suppose that there exists a positive constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \geq \theta \sum_{i=1}^{n} |\xi_i|^2, \tag{4.4}$$

for all $x \in \Omega$. and for all $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n$. We shall consider uniformly elliptic second order differential operators in divergence form, that is, operators that derive from differential expressions of the type

$$Su = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + c(x)u, \tag{4.5}$$

whose coefficients satisfy the aforementioned conditions. In addition we shall require that, $c \in L^\infty(\Omega)$ and for some positive constant c

$$c(x) \geq c \tag{4.6}$$

for almost every $x \in \Omega$. With the operator S we shall consider the homogenous Dirichlet boundary conditions. The *maximal classical Dirichlet realization* in $L^2(\Omega)$ of the formal differential expression S in (4.5), which will be denoted by S_D (or $S_{\Omega,D}$ if we want to emphasize its dependence on the domain Ω), is defined in the following way:

$$\mathrm{D}(S_D) = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) \, : \, u|_{\partial\Omega} = 0 \quad \text{and} \quad Su \in L^2(\Omega)\}$$

and

$$S_D u = Su \quad \text{for} \quad u \in \mathrm{D}(S_D).$$

Clearly $C_c^\infty(\Omega) \subset \mathrm{D}(S_D)$, hence the operator S_D is densely defined in $L^2(\Omega)$.

The operator S is closable in $L^2(\Omega)$, and the graph space of its closure in $L^2(\Omega)$ is the completion of $C^2(\Omega) \cap C(\overline{\Omega})$ with respect to the norm

$$\|u\|_{H^S(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|Su\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}, \quad (u \in C^2(\Omega))$$

and is denoted by $H^S(\Omega)$.

We shall estimate the gap between two elliptic operators defined on different open sets Ω_1, Ω_2 of $\mathbb{R}^n$ such that

$$\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon,$$

where ε is an arbitrary but fixed positive number. The main focus here are domain perturbation problems and therefore we shall not take into consideration the eventual dependence of the coefficients of (4.5) from the domain. Thus, we assume that the coefficients $a_{i,j}, c$ are defined in all of $\mathbb{R}^n$ and satisfy the relative properties required of them in all of $\mathbb{R}^n$.

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ be open sets as above. In each of them is defined the elliptic operator $S_{\Omega_i,D}$, $i = 1, 2$. By Definition 3.2, the gap between these operators is:

$$\delta(S_{\Omega_2,D},\, S_{\Omega_1,D}) = \sup_{\substack{u \in \mathrm{D}(S_{\Omega_2,D}) \\ \|v\|_{H^S(\Omega_2)=1}}} \mathrm{dist}_{H^S(\Omega_2)} \left(v,\ \mathrm{D}(S_{\Omega_1,D} \oplus S_{\Omega_2 \setminus \overline{\Omega_1},D})\right), \tag{4.7}$$

where

$$\mathrm{dist}_{H^S(\Omega_2)} \left(v,\ \mathrm{D}(S_{\Omega_1,D} \oplus S_{\Omega_2 \setminus \overline{\Omega_1},D})\right) = \inf_{\substack{u \in C(\Omega_2) \\ u|_{\partial\Omega_1} = 0 \; u|_{\partial\Omega_2} = 0 \\ Su \in C^2(\Omega_1) \\ Su \in C^2(\Omega_2 \setminus \overline{\Omega_1})}} \mathcal{K}_S(u - v) \tag{4.8}$$

with

$$\mathcal{K}_S(u-v) = \Big(\|v-u\|^2_{L^2(\Omega_2)} + \|Sv - Su\|^2_{L^2(\Omega_1)} + \|Sv - Su\|^2_{L^2(\Omega_2\setminus\overline{\Omega_1})} \Big)^{\frac{1}{2}}.$$

In (4.8), if we make the transformation $w = v - u$ and we find that

$$\operatorname{dist}_{H^S(\Omega_2)}\Big(v,\ \mathrm{D}(S_{\Omega_1,D} \oplus S_{\Omega_2\setminus\overline{\Omega_1},D})\Big) = \inf_{\substack{w \in C(\Omega_2)\\ w|_{\partial\Omega_1} = v|_{\partial\Omega_1}\ w|_{\partial\Omega_2} = 0\\ Sw \in C^2(\Omega_1)\\ Sw \in C^2(\Omega_2\setminus\overline{\Omega_1})}} \mathcal{K}_S(w)$$

Proceeding in the same way as in Lemma 4.1, using a theorem on the classical solvability of the corresponding homogeneous equation for continuous boundary data and the maximum principle we find that

$$\operatorname{dist}_{H^S(\Omega_2)}\Big(v,\ \mathrm{D}(S_{\Omega_1,D} \oplus S_{\Omega_2\setminus\overline{\Omega_1},D})\Big) \leq (\text{meas } \Omega_2)^{\frac{1}{3}} \|v\|_{C(\partial\Omega_1)}. \tag{4.9}$$

Now let the dimension $n \leq 3$. Let

$$\gamma = \begin{cases} 1 & \text{if } n = 1,\\ \text{any number in } (0,1) & \text{if } n = 2,\\ \frac{1}{2} & \text{if } n = 3. \end{cases} \tag{4.10}$$

In the same fashion as in the proof of Lemma 4.2, using a $H^2(\Omega)$-estimate and a Sobolev imbedding theorem, we attain the estimate

$$\sup_{x\in\Omega_2} \operatorname{dist}(x, \partial\Omega_2)^{-\gamma}|v(x)| \leq M\|u\|_{H^S(\Omega_2)}, \tag{4.11}$$

where the constant $M > 0$ is independent of v.

Finally, using this last estimate as in Theorem 4.1 we attain the analogous result.

Theorem 4.2. *Let $n \leq 3$, $\varepsilon > 0$ as in* (4.10)*, Ω_1, Ω_2 be open bounded sets in $\mathbb{R}^n$ with $C^{1,r}$ boundaries, where $\frac{1}{2} < r < 1$, and such that $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$ and both Ω_1 and $\Omega_2 \setminus \overline{\Omega_1}$ satisfy the exterior sphere condition. Assume also that Ω_2 satisfies the cone property. Then*

$$\delta(S_{\Omega_2,D},\ S_{\Omega_1,D}) \leq M\varepsilon^\gamma, \tag{4.12}$$

where the constant $M > 0$ depends only on Ω_2, γ and the coefficients $(a_{i,j})_{i,j=1,\dots,n}$, c of the differential expression in (4.5).

5 Spectral stability estimates for the eigenfunctions of elliptic operators

Let $\Omega \subset \mathbb{R}^n$, as usual, be an non empty open set. The notation $S_{\Omega,D}$ will denote either the classical Dirichlet Laplacian or the general elliptic operator as defined in §4.2, Since the operator $S_{\Omega,D}$ is symmetric and non-negative in $L^2(\Omega)$, we can consider its Friedrich's extension $S^F_{\Omega,D}$. We have done this in Theorem 2.8 for the Dirichlet Laplacian, and repeat the construction here for the more general case of an elliptic operator.

Thus, with notations introduced in §4.2, we consider the non-negative sesquilinear form

$$Q : \mathrm{D}(S_{\Omega,D}) \times \mathrm{D}(S_{\Omega,D}) \to \mathbb{C} \tag{5.1}$$

defined by

$$Q(u,v) = (S_{\Omega,D}, v)_{L^2(\Omega)} = \int_\Omega \Big(\sum_{i,j=1}^n a_{ij}(x) u_{x_i}(x) \overline{v_{x_j}(x)} + c(x) u(x) \overline{v(x)} \Big) dx \tag{5.2}$$

for all $u, v \in \mathrm{D}(S_{\Omega,D})$. With a little abuse of notations, by Q will be denoted also the quadratic form associated to the above sesquilinear form. This quadratic form is closable and the assumptions on the coefficients imply that the norms $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_Q$ are equivalent to each other in $\mathrm{D}(S_{\Omega,D})$. Therefore the completion of $\mathrm{D}(S_{\Omega,D})$ endowed with the norm $\|\cdot\|_Q$ is $H^1_0(\Omega)$. Applying Theorem 2.2 to its closure $\bar{Q}$, we find that there exists a unique non-negative self-adjoint operator $S^F_{\Omega,D}$ that gives rise to $\bar{Q}$. More explicitly, we have

$$\mathrm{D}({S^F_{\Omega,D}}^{1/2}) = \mathrm{D}(Q) = H^1_0(\Omega)$$

and $u \in \mathrm{D}(S^F_{\Omega,D})$ if, and only if, $u \in H^1_0(\Omega)$ and there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \Big(\sum_{i,j=1}^n a_{ij}(x) u_{x_i}(x) \overline{v_{x_j}(x)} + c(x) u(x) \overline{v(x)} \Big) dx = \int_\Omega f(x) \overline{v(x)} dx$$

for all $v \in H^1_0(\Omega)$ (the derivatives here are to be understood in the weak sense if function v is not smooth), and in this case

$$H^F_D u = f.$$

The letter F in the above notation is used to indicate the fact that the operator $S^F_{\Omega,D}$ is just the Friedrich's extension (see Definition 2.4) of $S_{\Omega,D}$.

Since $(H^1_0(\Omega), \|\cdot\|_{{S^F_{\Omega,D}}^{1/2}})$ is compactly embedded in $L^2(\Omega)$ (observe that $\|\cdot\|_{{S^F_{\Omega,D}}^{1/2}} = \|\cdot\|_Q$) it follows, by Theorem 2.5 that ${S^F_{\Omega,D}}^{1/2}$ is an operator with compact resolvent. Then Theorem 2.7 implies that also $S^F_{\Omega,D}$ has also compact resolvent. Theorem 2.6 and Lemma 2.3 imply that ${S^F_{\Omega,D}}^{1/2}$ and $S^F_{\Omega,D}$ have the same eigenfunctions and eigenspaces. Moreover, $\lambda \geq 0$ is an eigenvalue of $S^F_{\Omega,D}$ of multiplicity m ($m \in \mathbb{N}$) if, and only if, $\sqrt{\lambda}$ is an eigenvalue of ${S^F_{\Omega,D}}^{1/2}$ of multiplicity m, and as we said

$$\mathrm{N}({S^F_{\Omega,D}}^{1/2} - \sqrt{\lambda}) = \mathrm{N}(S^F_{\Omega,D} - \lambda).$$

The spectrum of the operator $S^F_{\Omega,D}$ is discrete, that is, consists of isolated eigenvalues of finite multiplicity. These eigenvalues are all strictly positive and if arranged in a sequence in increasing order, then the said sequence is divergent. Moreover, $S^F_{\Omega,D}$ admits an orthonormal basis of eigenfunctions. Let us introduce some notations that will be useful in the sequel: we arrange the eigenvalues in increasing order,

$$\lambda_1[\Omega] \leq \lambda_2[\Omega] \leq \cdots \leq \lambda_k[\Omega] \leq \cdots, \tag{5.3}$$

each of the eigenvalues repeated according to its multiplicity. Let also $\{\varphi_k[\Omega]\}_{k\in\mathbb{N}}$ be an orthonormal basis of $L^2(\Omega)$, where for each $k \in \mathbb{N}$, $\varphi_k[\Omega]$ is an eigenfunction for the eigenvalue $\lambda_k[\Omega]$; and let $k, m \in \mathbb{N}$ be such that

$$\lambda_{k-1}[\Omega] < \lambda_k[\Omega] \leq \cdots \leq \lambda_{k+m-1}[\Omega] < \lambda_{k+m}[\Omega],$$

(the first inequality on the left is not present if $k = 1$) then we set

$$N_{k,\,m}[\Omega] = \operatorname{span}\{\varphi_k[\Omega], \ldots, \varphi_{k+m-1}[\Omega]\}. \tag{5.4}$$

We consider also the closure $\overline{S_{\Omega,D}}$ of S_Ω. If the domain Ω has sufficiently regular boundary $\partial\Omega$ or if its boundary satisfies certain geometric properties, then the operator is essentially self-adjoint, that is to say

$$S^F_{\Omega,D} = \overline{S_{\Omega,D}}.$$

If the dimension of the space $n \leq 3$ and the bounded open set Ω is assumed to have a $C^{1,r}$ boundary with $\frac{1}{2} < r < 1$, we have proved this assertion for the Dirichlet Laplacian in §2.6 (see in particular Theorem 2.9 and Theorem 2.10). Under the same assumptions the result can be extended to more general elliptic operators with homogenous Dirichlet boundary values derived from differential expressions of the kind (4.5).

From now and on we shall drop the subscript D from the operators $S_{\Omega,D}$, $\overline{S_{\Omega,D}}$, $S^F_{\Omega,D}$ and shall indicate them simply by S_Ω, $\overline{S_\Omega}$, S^F_Ω respectively because we are dealing only with Dirichlet boundary conditions and there is no danger of confusion.

Now let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^n$ be open sets. We shall make use of the following identification of Hilbert spaces:

$$L^2(\Omega_2) \cong L^2(\Omega_1) \oplus L^2(\Omega_2 \setminus \Omega_1).$$

In this context we have also the identification of operators:

$$S_{\Omega_1,\,\Omega_2} = S_{\Omega_1} \oplus S_{\Omega_2\setminus\overline{\Omega_1}},$$

where $S_{\Omega_1,\,\Omega_2}$ is defined as in §3.2.1. Therefore

$$\delta(S^F_{\Omega_1}, S^F_{\Omega_2}) = \delta(S_{\Omega_1}, S_{\Omega_2}) = \delta(S_{\Omega_1} \times S_{\Omega_2\setminus\overline{\Omega_1}},\, S_{\Omega_2})$$

and

$$\delta(S^F_{\Omega_2}, S^F_{\Omega_1}) = \delta(S_{\Omega_2}, S_{\Omega_1}) = \delta(S_{\Omega_2},\, S_{\Omega_1} \times S_{\Omega_2\setminus\overline{\Omega_1}}),$$

where the gap is being taken in $L^2(\Omega_2) = L^2(\Omega_1) \times L^2(\Omega_2 \setminus \Omega_1)$. In obtaining these relations we have made use of the fact that S_{Ω_1}, S_{Ω_2} are essentially self-adjoint, of (1.17); of the fact that with these identifications we have also that $S^F_{\Omega_1} \oplus S^F_{\Omega_2\setminus\overline{\Omega_1}}$

is the Friedrich's extension of $S_{\Omega_1} \oplus S_{\Omega_2 \setminus \overline{\Omega_1}} = S_{\Omega_1, \Omega_2}$ in $L^2(\Omega_2)$; yet, $S_{\Omega_1} \oplus S_{\Omega_2 \setminus \overline{\Omega_1}}$ is closable in $L^2(\Omega_2)$ and its closure is $\overline{S_{\Omega_1}} \oplus \overline{S_{\Omega_2 \setminus \overline{\Omega_1}}}$ (see §2.7).

Let $\varepsilon > 0$ and assume that Ω_1, Ω_2 satisfy the hypotheses of Theorem 4.2. Then, by Theorem 4.1, if the dimension is $n \leq 3$,

$$\delta(S^F_{\Omega_1} \oplus S^F_{\Omega_2 \setminus \overline{\Omega_1}}, S^F_{\Omega_2}) = \delta(S_{\Omega_2}, S_{\Omega_2 \setminus \overline{\Omega_1}}) \leq M\varepsilon^\gamma,$$

where $M > 0$ is some constant independent of ε and of Ω_1; here $\gamma = 1/2$ if $n = 3$ while γ is any number in $(0, 1)$ if $n = 2$.

Let

$$\lambda_{\min}(S^F_{\Omega_2 \setminus \overline{\Omega_1}}) = \min \sigma(S^F_{\Omega_2 \setminus \overline{\Omega_1}})$$

be the smallest eigenvalue of $S^F_{\Omega_2 \setminus \overline{\Omega_1}}$ which is given by one of the variational or Rayleigh formulae (see [20, §4.5]);

$$\lambda_{\min}(S^F_{\Omega_2 \setminus \overline{\Omega_1}}) = \inf_{\substack{u \in H^1_0(\Omega_2 \setminus \overline{\Omega_1}) \\ u \neq 0}} \frac{\bar{Q}(u, u)}{\int_{\Omega_2 \setminus \Omega_1} |u|^2 dx}.$$

Using a Poincaré type inequality[19] assuming that Ω_2 is a set of finite measure, we find that

$$\lambda_{\min}(S^F_{\Omega_2 \setminus \overline{\Omega_1}}) \geq \frac{c_n \theta}{\operatorname{meas}(\Omega_2 \setminus \Omega_1)^{\frac{2}{n}}}, \tag{5.5}$$

where $c_n > 0$ is a constant that depends only on the dimension n and $\theta > 0$ is the constant of ellipticity (in the case of the Laplacian $\theta = 1$).

With these considerations we can prove the following theorem:

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[19] For instance, in [23, p. 263] is proved the so-called Gagliardo-Nirenberg-Sobolev inequality, namely, if $n \in \mathbb{N}$, $1 \leq p < n$, then

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq c_{n,p} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

for all $u \in C^\infty_c(\mathbb{R}^n)$ (and hence for all $u \in W^{1,p}_0(\Omega)$, Ω is an arbitrary open set), where $c_{n,p} = p(n-1)/(n-p)$ and p^* is such that $1/p^* = 1/p - 1/n$. In our case $p = 2$. If $n > 2$, $2 < 2^* = 2n/(n-2)$ and using Hölder inequality and the above-mentioned inequality, we have

$$\|u\|_{L^2(\Omega)} \leq (\operatorname{meas} \Omega)^{\frac{1}{2} - \frac{1}{2^*}} \|u\|_{L^{2^*}(\Omega)} \leq c_{n,2} (\operatorname{meas} \Omega)^{\frac{1}{n}} \|\nabla u\|_{L^2(\Omega)}$$

for all $u \in H^1_0(\Omega)$.

If $n = 2$ choose $p_0 < 2$ such that ${p_0}^* > 2$. By Hölder and Gagliardo-Nirenberg-Sobolev inequalities it follows that

$$\begin{aligned} \|u\|_{L^2(\Omega)} &\leq (\operatorname{meas} \Omega)^{\frac{1}{2} - \frac{1}{{p_0}^*}} \|u\|_{L^{{p_0}^*}(\Omega)} \leq c_{n,p_0} (\operatorname{meas} \Omega)^{\frac{1}{2} - \frac{1}{{p_0}^*}} \|\nabla u\|_{L^{p_0}(\Omega)} \\ &\leq c_{n,p_0} (\operatorname{meas} \Omega)^{\frac{1}{2} - \frac{1}{{p_0}^*} + \frac{1}{p_0} - \frac{1}{2}} \|\nabla u\|_{L^2(\Omega)} = c_{n,p_0} (\operatorname{meas} \Omega)^{\frac{1}{n}} \|\nabla u\|_{L^2(\Omega)} \end{aligned}$$

for all $u \in H^1_0(\Omega)$.

It remains the case $n = 1$. Let $\Omega = (a, b)$ be an interval and $u \in H^1_0(a, b)$. Then, by Hölder inequality,

$$\int_a^b |u(x)|^2 dx = \int_a^b \left| \int_a^x u'(t) dt \right|^2 dx \leq \frac{(b-a)^2}{2} \int_a^b |u'(x)|^2 dx.$$

If Ω is an arbitrary open set of $\mathbb{R}$, the result is attained by applying the previous inequality to each of its connected components.

Theorem 5.1. *Let $n \leq 3$, $\gamma = 1/2$ if $n = 3$, $0 < \gamma < 1$ if $n = 2$ and $\gamma = 1$ if $n = 1$. Let Ω_2 be a bounded open set of $\mathbb{R}^n$. Assume that $\Omega_2 \in C^{1,r}$ ($\frac{1}{2} < r < 1$) and satisfies the exterior sphere condition. Let $\lambda[\Omega_2]$ be an eigenvalue of multiplicity m ($m \in \mathbb{N}$) of $S^F_{\Omega_2}$, that is, there exists $k \in \mathbb{N}$ such that $\lambda[\Omega_1] = \lambda_k[\Omega_2] = \cdots = \lambda_{k+m-1}[\Omega_2]$. Then there exist $M_k, \varepsilon_k > 0$ such that the following holds: if $0 < \varepsilon < \varepsilon_k$ and $\Omega_1 \in C^{1,r}$ is such that $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$, and $\Omega_2 \setminus \overline{\Omega_1}$ satisfies the exterior sphere condition, then*

$$\hat{\delta}(N_{k,m}[\Omega_1],\ N_{k,m}[\Omega_2]) \leq M_k \varepsilon^\gamma. \tag{5.6}$$

Proof. Let the circle of radius $r > 0$ centered at $\lambda[\Omega_2]$ be such that the rest of the points of $\sigma(S^F_{\Omega_2})$ have distance greater than $2r$ from $\lambda[\Omega_2]$. We take $\varepsilon_k > 0$ such that $M{\varepsilon_k}^\gamma$ of estimate (4.12) is less than δ of Theorem 1.11, where $\lambda_0 = \lambda[\Omega_2]$; and $\lambda[\Omega_2]$ is less than the right side member of (5.5) when Ω_1 is replaced by[20] $(\Omega_2)_{\varepsilon_k}$. The second bound on ε_k is imposed to guarantee that $\lambda[\Omega_2]$ is not an eigenvalue of $S^F_{\Omega_2\setminus\overline{\Omega_1}}$, so that $\mathrm{N}(S^F_{\Omega_1} - \lambda[\Omega_1]) = \mathrm{N}(S^F_{\Omega_1} \oplus S^F_{\Omega_2\setminus\overline{\Omega_1}} - \lambda[\Omega_1])$ with the usual identifications (see §2.7). Then we apply Theorem 1.11, Lemma 1.3 and (4.2) to draw the conclusions of this theorem. □

Theorem 5.2. *Let $n \leq 3$, $\gamma = 1/2$ if $n = 3$, $0 < \gamma < 1$ if $n = 2$, and $\gamma = 1$ if $n = 1$. Let Ω_2 be an open set of $\mathbb{R}^n$. Assume that $\Omega_2 \in C^{1,r}$ for some $\frac{1}{2} < r < 1$ and satisfies the exterior sphere condition. Let $\lambda[\Omega_2]$ be an eigenvalue of multiplicity m ($m \in \mathbb{N}$) of $S^F_{\Omega_2}$, that is, there exists $k \in \mathbb{N}$ such that $\lambda[\Omega_2] = \lambda_k[\Omega_2] = \cdots = \lambda_{k+m-1}[\Omega_2]$. Then there exist $c_k, \varepsilon_k > 0$ such that the following holds: if $0 < \varepsilon < \varepsilon_k$ and $\Omega_1 \in C^{1,r}$ is such that $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$ and $\Omega_2 \setminus \overline{\Omega_1}$ satisfies the exterior sphere condition, and $\varphi_{k+1}[\Omega_1], \ldots, \varphi_{k+m}[\Omega_1]$ is an orthonormal set of eigenfunctions of $S^F_{\Omega_1}$ corresponding to the eigenvalues $\lambda_k[\Omega_1] \leq \cdots \leq \lambda_{k+m-1}[\Omega_1]$, then there exists an orthonormal set of eigenfunctions $\varphi_k[\Omega_2], \ldots, \varphi_{k+m-1}[\Omega_2]$ of $S^F_{\Omega_2}$ corresponding to the eigenvalue $\lambda[\Omega_2]$ of $S^F_{\Omega_2}$ such that*

$$\|\varphi_{k+i}[\Omega_1] - \varphi_{k+i}[\Omega_2]\|_{L^2(\Omega_2)} \leq c_k \varepsilon^\gamma \tag{5.7}$$

for all $i = 0, \ldots, m-1$.

Remark 5.1. *A natural question is the following: What are the minimal assumptions on the smoothness of the boundaries of the open sets Ω_i, ($i = 1, 2$) where the above-mentioned elliptic operators are defined for the previous theorems to continue to be valid? Retracing the proofs we find out that these assumptions are those that guarantee first, the essential self-adjointness of the said operators, that is, $\mathrm{D}(S_{\Omega_i,D})$ is dense in $\mathrm{D}(S^F_{\Omega_i,D})$ w.r.t. the graph norm $\|\cdot\|_{H^s(\Omega_i)}$; second, the solvability of the homogeneous equation $Su = 0$ with continuous boundary data and the validity of the maximum principle, third, the global $H^2(\Omega_i)$-estimate for solutions of the equation $Su = f$ for zero Dirichlet boundary conditions and the validity of the imbedding $H^2(\Omega_i) \subset C^\gamma(\Omega_i)$, ($i = 1, 2$).*

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[20]By definition, given a set $G \subset \mathbb{R}^n$ and $\varepsilon > 0$,

$$G_\varepsilon = \{x \in G \ : \ \mathrm{dist}(x, \mathbb{R}^n \setminus G) > \varepsilon\};$$

$$G^\varepsilon = \{x \in \mathbb{R}^n \ : \ \mathrm{dist}(x, G) < \varepsilon\}.$$

Remark 5.2. *Estimate* (1.55) *can be used to deduce bounds on the rate of convergence of eigenvalues. However, the results that we obtain are not optimal, while sharp estimates are already present in literature.*

Remark 5.3. *The estimates of Theorem 1.9 can be used to study the dependence of solutions of elliptic equations on domain perturbation. Thus, we find that, if* $f \in L^2(\Omega_2)$ *and* $u_i \in L^2(\Omega_i)$ *is a weak solution of the problem*

$$\begin{cases} Su_i = f & \text{in } \Omega_i \\ u_i|_{\partial\Omega_i} = 0 & \text{on } \partial\Omega_i \end{cases},$$

$i = 1, 2,$ *then*

$$\|u_1 - u_2\|_{L^2(\Omega_1)} \leq C\varepsilon^\gamma \|f\|_{L^2(\Omega_2)},$$

if $0 < \varepsilon < \varepsilon_0$, $\Omega_1 \subset \Omega_2 \subset (\Omega_1)^\varepsilon$, Ω_1, Ω_2 *have* $C^{1,r}$ *boundaries,* Ω_1, $\Omega_2 \setminus \overline{\Omega_1}$ *satisfy the exterior sphere condition, and* $C, \varepsilon_0 > 0$ *are constants that depend only on the set* Ω_2 *and the coefficients of the differential operator* S.

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