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# Extension of the notion of a gap to differential operators defined on different open sets

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*The paper is dedicated to Professor Hans Triebel on the occasion of his 75th birthday*

In this paper the notion of a gap between two linear operators is extended to the case of linear differential operators defined on different open sets. Estimates of the gap between second order uniformly elliptic partial differential operators subject to homogeneous Dirichlet boundary conditions defined on different open sets $\Omega_1$ and $\Omega_2$ are obtained in terms of the geometrical characteristics of vicinity of $\Omega_1$ and $\Omega_2$. These estimates can be used for obtaining spectral stability estimates for the eigenvalues and eigenfunctions of the aforementioned operators.



## 1 Introduction

The study of perturbation problems for linear operators necessitates a notion of "vicinity" or "proximity" between linear operators so that the "smallness" of a perturbation might be quantified. For this purpose a quantitative notion of a *gap*, *opening* or *aperture* has long been known. The notion of a gap is first introduced for the linear subspaces of a normed space and then is extended to linear operators acting between normed spaces by considering their graphs (see Section 2).

The notion of a gap between linear subspaces and between linear operators was first introduced by Krein and his co-workers in the 1940s (e.g., see [9], [10]). For the convenience of the reader the definition and some properties of gaps are given in Section 2.

Other metrics on the set of closed linear subspaces of a Banach space and, consequently, metrics on the set of closed linear operators acting between Banach spaces, have been introduced (see, for example, Massera, Schäffer [12], Newburgh [16], Berkson [1]).

The purpose of this paper is to extend the definition of a gap to a class of linear operators defined on different normed spaces. This is done in Section 3. Then estimates of the gap for certain second order elliptic operators defined on different open sets are obtained in Section 4.

The standard notion of a gap appeared to be quite useful in a number of perturbation problems for linear operators (see, for example, Mennicken and Sagraloff [13]–[15]) and especially in the problem of spectral stability (see detailed exposition in Kato [8]). The definition given in this paper and the estimates for gaps are expected to be efficiently used for the same purposes in the case of linear differential operators defined on different open sets.

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## 2 Preliminaries and notation

Here the definition of a gap is given for arbitrary linear subspaces of a normed space, that is, we do not restrict ourselves to closed subspaces or closed linear operators as it is usually done in the literature. Proofs of all facts reported here about gaps between closed subspaces and closed operators can be found, for instance, in Kato's monograph [8]. The extension of these results to arbitrary linear subspaces or linear operators requires only slight modifications in the proofs; one should apply results obtained for closed subspaces or closed operators to the closures of subspaces or of graphs of operators.

### 2.1 Gap between subspaces

**Definition 2.1** The *gap* between two linear subspaces $M$ and $N$ of a normed space $Z$ is defined by the following formula:

$$\delta(M,N) = \sup_{\substack{u \in M \\ \|u\| = 1}} \operatorname{dist}(u,N), \tag{2.1}$$

where $\operatorname{dist}(u,N) = \inf_{v\in N} \|u-v\|$ is the distance of the vector $u$ to the subspace $N$. (The equality $\|u\| = 1$ can be replaced by the inequality $\|u\| \le 1$ without changing the result.)

One also defines

$$\hat{\delta}(M,\, N) = \max\{\delta(M,N),\ \delta(N,M)\}. \tag{2.2}$$

Sometimes the latter is called the *symmetric* or *maximal gap* between $M$ and $N$ to distinguish it from the former.

The supremum in (2.1) has no meaning if $M = \{0\}$; in this case one defines $\delta(\{0\},N) = 0$ for any $N$. On the other hand $\delta(M,\{0\}) = 1$ if $M \ne \{0\}$ as is seen by the definition.

The gap $\delta(M,N)$ can be characterized as the smallest number $\delta$ such that

$$\operatorname{dist}(u,N) \le \delta\|u\| \tag{2.3}$$

for all $u \in M$.

**Examples 2.2** (i) Let $Z = \mathbb{R}^2$ or $Z = \mathbb{R}^3$ and let $M$ and $N$ be two straight lines passing through the origin such that the angle between them is equal to $\alpha$. Then $\delta(M,N) = \delta(N,M) = \sin\alpha$.

(ii) Let $Z = \mathbb{R}^3$ and let $M$ and $N$ be two planes passing through the origin such that the angle between them is equal to $\alpha$. Then again $\delta(M,N) = \delta(N,M) = \sin\alpha$.

(iii) Let $Z = \mathbb{R}^3$, $M$ be a straight line passing through the origin and $N$ be a plane passing through the origin such that the angle between them is equal to $\alpha$. Then $\delta(M,N) = \sin\alpha$ (in particular, if $M \subset N$ then $\delta(M,N) = 0$), but $\delta(N,M) = 1$.

The following properties of a gap follow directly by the definition.

**Proposition 2.3** *Let $M$ and $N$ be two linear subspaces of a normed space $Z$. Then*

$$\delta(M,N) = \delta(\overline{M},\overline{N}), \quad \hat{\delta}(M,N) = \hat{\delta}(\overline{M},\overline{N}); \tag{2.4}$$

$$\delta(M,N) = 0 \quad \textit{if, and only if,} \quad \overline{M} \subset \overline{N}; \tag{2.5}$$

$$\hat{\delta}(M,N) = 0 \quad \textit{if, and only if,} \quad \overline{M} = \overline{N}; \tag{2.6}$$

$$\hat{\delta}(M,N) = \hat{\delta}(N,M); \tag{2.7}$$

$$0 \le \delta(M,N) \le 1, \quad 0 \le \hat{\delta}(M,N) \le 1. \tag{2.8}$$

If $Z$ is a normed space, the symbol $Z^*$ denotes the adjoint space of $Z$. Let $E \subset Z$. The *annihilator* of $E$ in $Z^*$ is denoted by $E^0$. There is a simple relation of the gap between two vector subspaces and the gap between their annihilators as stated in the following

**Theorem 2.4** *For any linear subspaces $M$ and $N$ of a normed space $Z$*

$$\delta(M,N) = \delta\big(N^0, M^0\big), \quad \hat{\delta}(M,N) = \hat{\delta}\big(M^0, N^0\big). \tag{2.9}$$

Let $M$ be a linear subspace of a Hilbert space $Z$. We denote by $P_M$ the orthogonal projector onto the closure $\overline{M}$. With this notation we have $P_M = P_{\overline{M}}$.

**Lemma 2.5** *Let $M$, $N$ be linear subspaces of a Hilbert space $Z$. Then*

$$\delta(M,N) = \|(1-P_N)P_M\| = \|P_M(1-P_N)\|. \tag{2.10}$$

Proof. Since for any $u \in H$ $\operatorname{dist}(u,N) = \|u - P_N u\|$, we have

$$\begin{aligned} \delta(M,N) &= \sup_{u\in M,\ \|u\|\le 1} \|u - P_N u\| = \sup_{u\in M,\ \|u\|\le 1} \|(1-P_N)P_M u\| \\ &= \sup_{u\in Z,\ \|u\|\le 1} \|(1-P_N)P_M u\| = \|(1-P_N)P_M\|. \end{aligned}$$

We also note that

$$\|P_M(1-P_N)\| = \|(P_M(1-P_N))^*\| = \big\|(1-P_N)^* P_M^*\big\| = \|(1-P_N)P_M\|. \qquad \square$$

**Theorem 2.6** *If $M$ and $N$ are linear subspaces of a Hilbert space $Z$, then*

$$\hat{\delta}(M,N) = \|P_M - P_N\|. \tag{2.11}$$

Proof. A proof of this statement, based on some other facts, is given in [8]. Here we give a direct proof. Let $\delta = \hat{\delta}(M,N)$. By Lemma 2.5

$$\delta = \max\{\|(1-P_N)P_M\|,\ \|P_N(1-P_M)\|\}.$$

Let $u \in Z$. Keeping in mind that ${P_M}^2 = P_M$, $(1-P_M)^2 = 1 - P_M$ and using the Pythagorean Theorem, we have

$$\begin{aligned} \|(P_M - P_N)u\|^2 &= \|(1-P_N)P_M u - P_N(1-P_M)u\|^2 \\ &= \|(1-P_N)P_M u\|^2 + \|P_N(1-P_M)u\|^2 \\ &\le \|(1-P_N)P_M\|^2\|P_M u\|^2 + \|P_N(1-P_M)\|^2\|(1-P_M)u\|^2 \\ &\le \delta^2\|P_M u\|^2 + \delta^2\|(1-P_M)u\|^2 \\ &= \delta^2\|u\|^2. \end{aligned}$$

Thus we have

$$\|(1-P_N)P_M u\|, \quad \|P_N(1-P_M)u\| \le \|(P_M - P_N)u\| \le \delta\|u\|$$

for every $u \in Z$, which, when taking the supremum over all of $u \in Z$ with $\|u\| \le 1$, yields

$$\delta = \max\{\|(1-P_N)P_M\|,\ \|P_N(1-P_M)\|\} \le \|P_M - P_N\| \le \delta,$$

which is exactly what we wanted to prove. $\square$

### 2.2 Gap between operators

The Cartesian product $X \times Y$ of two normed spaces $X$, $Y$ is a normed space with the usual definition of addition, multiplication by scalars and the norm defined by

$$\|(u,v)\|_{X\times Y} = \big(\|u\|_X^2 + \|v\|_Y^2\big)^{1/2}$$

 

for all $u \in X$, $v \in Y$. If $X$ and $Y$ are inner product spaces with the inner products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$ respectively, then $X \times Y$ is also an inner product space with the inner product defined by

$$((u,v),(u',v'))_{X\times Y} = (u,u')_X + (v,v')_Y$$

for all $u, u' \in X$, $v, v' \in Y$. Thus, if $X, Y$ are Banach (Hilbert) spaces, then $X \times Y$ is also a Banach (Hilbert) space. As usual, the subscripts in the notation for norms and inner products will be dropped when there is no ambiguity.

Let

$$T : D(T) \subset X \longrightarrow Y$$

be a linear operator, where the linear subspace $D(T)$ of $X$ is the domain of the operator $T$. The kernel of the operator $T$ is denoted by $N(T)$ and its range by $R(T)$. The linear subspace

$$G(T) = \{(u, Tu) : u \in D(T)\}$$

of $X \times Y$ is by definition the graph of the operator $T$.

**Definition 2.7** Let $X$ and $Y$ be normed spaces and let

$$S : D(S) \subset X \longrightarrow Y, \quad T : D(T) \subset X \longrightarrow Y$$

be linear operators acting from $X$ to $Y$. The *gap between $S$ and $T$* is defined by

$$\delta(S,T) = \delta(G(S), G(T)), \tag{2.12}$$

and the *symmetric* or *maximal gap* between $S$ and $T$ by

$$\hat{\delta}(S,T) = \hat{\delta}(G(S), G(T)). \tag{2.13}$$

More explicitly,

$$\delta(S,T) = \sup_{\substack{u \in D(S) \\ \|u\|_X^2 + \|Su\|_Y^2 = 1}} \inf_{v\in D(T)} \left(\|u-v\|_X^2 + \|Su - Tv\|_Y^2\right)^{1/2}. \tag{2.14}$$

The equality $\|u\|_X^2 + \|Su\|_Y^2 = 1$ can be replaced by the inequality $\|u\|_X^2 + \|Su\|_Y^2 \le 1$ in the right-hand side of (2.14) without changing the result.

If $S$ and $T$ are closable operators with closures $\overline{S}$ and $\overline{T}$, then

$$\delta(S,T) = \delta\big(\overline{S}, \overline{T}\big). \tag{2.15}$$

**Theorem 2.8** *Let $X, Y$ be normed spaces and let $S$, $T$ be densely defined linear operators. Then*

$$\delta(S,\, T) = \delta(T^*,\, S^*), \quad \hat{\delta}(S,\, T) = \hat{\delta}(S^*,\, T^*), \tag{2.16}$$

*where $S^*$, $T^*$ are the adjoint operators of $S$ and $T$ respectively.*

**Corollary 2.9** *Let $X$ be a Hilbert space, and let $S, T$ be essentially self-adjoint* (*in particular, self-adjoint*) *linear operators in $X$. Then*

$$\delta(S,T) = \delta(T,S) = \hat{\delta}(S,T). \tag{2.17}$$

**Example 2.10** Let $T$ be a bounded linear operator with domain $D(T) = X$. It is easily seen that

$$\delta(T,0) = \frac{\|T\|}{(1+\|T\|^2)^{1/2}}.$$

In virtue of (2.16) it follows also that $\delta(0,T) = \delta(T^*,0) = \delta(T,0) = \hat{\delta}(T,0)$. Further results for bounded operators can be found in [3].

### 2.3 Direct sum of operators

Let $X,\ X'$ be normed spaces. Any subspace $M \subset X$ can be seen as a subspace of $X \times X'$ by identifying it with

$$\{(u, 0)\ :\ u \in M\};$$

analogously any subspace $M' \subset X'$ will be identified with the subspace

$$\{(0, u')\ :\ u' \in M'\}$$

of $X \times X'$. If $u \in X$ and $u' \in X'$, we see $u$ and $u'$ as elements of $X \times X'$ by identifying them with $(u, 0)$ and $(0, u')$ respectively. Hence

$$(u,\ u') = (u,\ 0) + (0,\ u') \equiv u + u'.$$

With this convention we have $X \times X' = X \dot{+} X'$ and

$$\|(u, u')\|_{X \times X'} = \|(u, 0) + (0, u')\|_{X \dot{+} X'} = \left(\|u\|_X^2 + \|u\|_{X'}^2\right)^{1/2}.$$

**Definition 2.11** Let $X,\ X',\ Y,\ Y'$ be normed spaces and let

$$S : D(S) \subset X \longrightarrow Y, \quad S' : D(S') \subset X' \longrightarrow Y'$$

be linear operators. Then the *direct sum operator* $S \dot{+} S'$ of $S$ and $S'$ is defined in the following way:

$$D(S \dot{+} S') = D(S) \dot{+} D(S'),$$

and

$$(S \dot{+} S')(u + u') = Su + S'u'$$

for all $u \in D(S),\ u' \in D(S')$.

If $X,\ X',\ Y,\ Y'$ are Hilbert spaces we write $S \oplus S'$ instead of $S \dot{+} S'$. In the next proposition some easy-to-prove properties of the direct sum operator are listed.

**Proposition 2.12**

(i)

$$N(S \dot{+} S') = N(S) \dot{+} N(S'), \quad R(S \dot{+} S') = R(S) \dot{+} R(S').$$

(ii)

$$G(S \dot{+} S') = G(S) \dot{+} G(S').$$

(iii) *$S \dot{+} S'$ is invertible if, and only if, $S,\ S'$ are both invertible and, in this case,*

$$(S \dot{+} S')^{-1} = S^{-1} \dot{+} S'^{-1}.$$

(iv) *$S \dot{+} S'$ is a closable operator if, and only if, $S$, $S'$ are closable operators and,*

$$\overline{S \dot{+} S'} = \overline{S} \dot{+} \overline{S'};$$

*in particular, if $S, S'$ are closed operators, then so is $S \dot{+} S'$.*

(v) *$S \dot{+} S'$ is a densely defined operator if, and only if, $S$, $S'$ are densely defined operators and in this case*

$$(S \dot{+} S')^* = S^* \dot{+} S'^*.$$

(vi) *$S \dot{+} S'$ is a bounded (respectively, compact) operator if, and only if, both $S$ and $S'$ are bounded (respectively, compact) operators.*

*Next let $X = Y,\ X' = Y'$ be Hilbert spaces.*

(vii) *$S \oplus S'$ is self-adjoint (essentially self-adjoint) if, and only if, both $S,\ S'$ are self-adjoint (essentially self-adjoint) operators.*

(viii) *If $S,\ S'$ are non-negative symmetric densely defined linear operators and $S^F,\ {S'}^F$ are their Friedrich extensions (see, for example,* [4, Section 4.4]*), then*

$$(S \oplus S')^F = S^F \oplus S'^F.$$

 

### 2.4 The Dirichlet Laplacian

First, we recall briefly some standard notation about function spaces. Given an open set $\Omega \subset \mathbb{R}^n$ and $l \in \mathbb{N}$. as usual, let $H^l(\Omega)$ denote the Sobolev space equipped with the norm

$$\|u\|_{H^l(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \left\|\nabla^l_w u\right\|^2_{L^2(\Omega)}\right)^{1/2},$$

where $\nabla^l_w u$ is the weak or distributional gradient of order $l$ of $u$. Let also $H^l_0(\Omega)$ denote the closure of the space $C^\infty_c(\Omega)$ of all infinitely continuously differerentiable functions with compact supports contained in $\Omega$ with respect to the norm of $H^l(\Omega)$.

For any open set $\Omega \subset \mathbb{R}^n$ we consider the *(generalized) Dirichlet Laplacian*

$$-\Delta_{D,\Omega} : D(-\Delta_{D,\Omega}) \subset L^2(\Omega) \longrightarrow L^2(\Omega)$$

defined via the Friedrich's extension procedure in the following way: $u \in D(-\Delta_{D,\Omega})$ if, and only if $u \in H^1_0(\Omega)$ and there exists $f \in L^2(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \overline{\nabla v}\, dx = \int_\Omega f \overline{v}\, dx$$

for all $v \in H^1_0(\Omega)$; in this case one defines

$$-\Delta_{D,\Omega} u = f.$$

Let $\Omega$ be an open set in $\mathbb{R}^n$ and let $u, v \in L^1_{\mathrm{loc}}(\Omega)$. The function $v$ is the *weak Laplacian* of $u$ on $\Omega$ (briefly $v = \Delta_w u$) if for all $\phi \in C^\infty_c(\Omega)$

$$\int_\Omega u \Delta\phi\, dx = \int_\Omega v\phi\, dx\,.$$

Let

$$H^\Delta(\Omega) = \left\{u \in L^2(\Omega) \,:\, \exists \Delta_w u \in L^2(\Omega)\right\}$$

be the *Laplacian graph space* in $L^2(\Omega)$. with the norm

$$\|u\|_{H^\Delta(\Omega)} = \left(\|u\|^2_{L^2(\Omega)} + \|\Delta_w u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

The Dirichlet Laplacian can be described in the following way:

$$D(-\Delta_{D,\Omega}) = H^1_0(\Omega) \cap H^\Delta(\Omega),$$

and

$$(-\Delta_{D,\Omega})u = -\Delta_w u$$

for all $u \in D(-\Delta_{D,\Omega})$.

Under additional regularity assumptions on the boundary of the open set $\Omega$ the domain of the Dirichlet Laplacian can be described better.

We need to recall the definition of an open set having a $C^{1,r}$ boundary, $0 < r \le 1$. Let $\mathcal{N}$ be a space of real-valued functions in $\mathbb{R}^{n-1}$ ($n \in \mathbb{N}$, $n \ge 2$). One says that a bounded open set $\Omega \subset \mathbb{R}^n$ *has a boundary of class* $\mathcal{N}$ or simply *a* $\mathcal{N}$ *boundary* (we write also $\partial\Omega \in \mathcal{N}$) if there exists a finite open covering $\{V_j\}_{1 \le j \le N}$ of the boundary $\partial\Omega$ of $\Omega$ with the property that, for every $j \in \{1, \ldots, N\}$, $V_j \cap \Omega$ coincides with the portion lying in the over-graph of a function $\varphi_j \in \mathcal{N}$ (considered in a new system of coordinates from the original one via a rigid motion). Two special cases are going to play a particularly important role in the sequel. First, if $\mathcal{N} = \mathrm{Lip}\left(\mathbb{R}^{n-1}\right)$, the space of real-valued functions satisfying a (global) Lipschitz condition in $\mathbb{R}^{n-1}$, we shall refer to $\Omega$ as *having a Lipschitz boundary*. Second, corresponding to the case when $\mathcal{N}$ is the subspace of $\mathrm{Lip}\left(\mathbb{R}^{n-1}\right)$ consisting of functions whose first-order derivatives satisfy a (global) Hölder condition of order $r \in\, ]0, 1]$, we shall say that $\Omega$ *has a* $C^{1,r}$ *boundary*.

**Theorem 2.13** *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and assume that $\partial\Omega \in C^{1,r}$ for some $1/2 < r \le 1$. Then*

$$D(-\Delta_{D,\,\Omega}) = H^1_0(\Omega) \cap H^2(\Omega).$$

P r o o f. It suffices to prove that $D(-\Delta_{D,\,\Omega}) \subset H^2(\Omega)$. If $\partial\Omega \in C^{1,1}$ this inclusion is a consequence of the a priori $L^2$-estimates for solutions to elliptic equations. This classical result can be found, for instance, in [6, Theorem 9.15, Section 9.5, p. 239]. For the more general case of $\partial\Omega \in C^{1,r}$ with $1/2 < r \le 1$ see [5, Lemma A.1]. □

The previous result holds also for a convex open set $\Omega$ as proved by Kadlec in [7].

**Theorem 2.14** *Let $n = 2, 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded open set. Assume that $\partial\Omega \in C^{1,r}$ with $1/2 < r \le 1$. Then*[1]

$$D(-\Delta_{D,\,\Omega}) = \overline{\{u \in C^2(\Omega) \cap H^2(\Omega)\ :\ u|_{\partial\Omega} = 0\}},$$

*where the closure is in $H^2(\Omega)$.*

P r o o f. Since $n \le 3$, by the Sobolev embedding theorem each function $u \in H^1_0(\Omega) \cap H^2(\Omega)$ is equivalent to a function $g$ belonging to $C(\overline{\Omega})$. Since $u \in H^1_0(\Omega)$ it has zero trace at $\partial\Omega$, for example, in the sense of [2, Chapter 5]. Therefore $g|_{\partial\Omega} = 0$. By Theorem 3 and Corollary 2 of Chapter 2 of [2] it follows that there exists a sequence $\{g_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega) \cap C(\overline{\Omega}) \cap H^2(\Omega)$ such that $g_k \to g$ in $C(\overline{\Omega})$ as $k \to \infty$ and $g_k|_{\partial\Omega} = 0$ for all $k \in \mathbb{N}$. Moreover, by the proofs of Theorems 2–3 and Corollary 2 of Chapter 2 of [2] it follows that the aforementioned sequence can be chosen in such a way that also $g_k \to u$ in $H^2(\Omega)$. □

## 3 Gaps between operators defined on different open sets

### 3.1 Definition and examples

In the present section we propose an extension of the notion of a gap for operators defined on different open sets.

Given two open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ and linear operators

$$S_{\Omega_i} : D(S_{\Omega_i}) \subset X(\Omega_i) \longrightarrow Y(\Omega_i),$$

$i = 1, 2$, where $X(\Omega_i)$ and $Y(\Omega_i)$ are normed spaces of functions defined on $\Omega_i$, we aim at considering extensions $X(\Omega_1 \cup \Omega_2)$[2] of the spaces $X(\Omega_1)$ and $X(\Omega_2)$, $Y(\Omega_1 \cup \Omega_2)$ of the spaces $Y(\Omega_1)$ and $Y(\Omega_2)$ respectively, constructing certain extensions

$$\tilde{S}_{\Omega_i} : D\big(\tilde{S}_{\Omega_i}\big) \subset X(\Omega_1 \cup \Omega_2) \longrightarrow Y(\Omega_1 \cup \Omega_2)$$

of the operators $S_{\Omega_i}$ for which $D\big(\tilde{S}_{\Omega_i}\big)|_{\Omega_i} = D(S_{\Omega_i})$, and defining the gap between the operators $S_{\Omega_1}$ and $S_{\Omega_2}$ as the gap between the operators $\tilde{S}_{\Omega_1}$ and $\tilde{S}_{\Omega_2}$ whose domains lie in the same space $X(\Omega_1 \cup \Omega_2)$.

Such definition of a gap depends, of course, on the way in which the spaces and the operators are extended. We choose the way of extending which is likely to have effective applications to the problem[3] of comparing eigenvalues and eigenfunctions of elliptic differential operators defined on different open sets.

We start with considering the general case.

We say that an open set $\Omega \subset \mathbb{R}^n$ is *regular* if meas $\partial\Omega = 0$.

Let with each regular open set $\Omega \subset \mathbb{R}^n$ normed spaces $X(\Omega)$, $Y(\Omega)$ and a linear operator

$$S_\Omega : D(S_\Omega) \subset X(\Omega) \longrightarrow Y(\Omega),$$

where $D(S_\Omega)$ is a linear subspace of $X(\Omega)$, be associated. We assume the the following assumptions are satisfied

[1] For all $x \in \partial\Omega$

$$u|_{\partial\Omega}(x) = \lim_{\substack{y \to x \\ y \in \Omega}} u(y).$$

[2] I.e., $X(\Omega_1 \cup \Omega_2)$ is a normed space of functions defined on $\Omega_1 \cup \Omega_2$ such that $X(\Omega_1 \cup \Omega_2)|_{\Omega_i} = X(\Omega_i)$, $i = 1, 2$.

[3] Applications to this problem will be considered in a separate paper.

 

(1) $X(\emptyset) = Y(\emptyset) = \{0\}$,

(2) for any regular open set $\Omega \subset \mathbb{R}^n$

$$X\big((\overline{\Omega})_0\big) \simeq X(\Omega), \quad Y\big((\overline{\Omega})_0\big) \simeq Y(\Omega),$$

where $A_0$ is the set of all inner points of a set $A \subset \mathbb{R}^n$,

(3) if $\Omega_1, \Omega_2$ are disjoint regular open sets, then

$$X(\Omega_1) \dot{+} X(\Omega_2) \simeq X(\Omega_1 \cup \Omega_2) \quad \text{and} \quad Y(\Omega_1) \dot{+} Y(\Omega_2) \simeq Y(\Omega_1 \cup \Omega_2). \tag{3.1}$$

Property (2) means that for any function $u \in X(\Omega)$ each function $v$, defined on $\big(\overline{\Omega}\big)_0$ and coinciding with $u$ on $\Omega$, belongs to $X\big(\big(\overline{\Omega}\big)_0\big)$ and $\|v\|_{X((\overline{\Omega})_0)} = \|u\|_{X(\Omega)}$. Conversely, for any function $v \in X\big(\big(\overline{\Omega}\big)_0\big)$ the function $v|_\Omega \in X(\Omega)$ and $\|v|_\Omega\|_{X(\Omega)} = \|v\|_{X((\overline{\Omega})_0)}$. The same refers to the second relation.

The first relation in (3.1) means that if $u_1 \in X(\Omega_1)$ and $u_2 \in X(\Omega_2)$, then

$$u = \begin{cases} u_1(x), & x \in \Omega_1 \\ u_2(x), & x \in \Omega_2 \end{cases} \in X(\Omega_1 \cup \Omega_2) \tag{3.2}$$

and

$$\|u\|_{X(\Omega_1 \cup \Omega_2)} = \big(\|u_1\|^2_{X(\Omega_1)} + \|u_2\|^2_{X(\Omega_2)}\big)^{1/2}.$$

Conversely, if $u \in X(\Omega_1 \cup \Omega_2)$, then $u|_{\Omega_1} \in X(\Omega_1)$, $u|_{\Omega_2} \in X(\Omega_2)$ and $\|u|_{\Omega_i}\|_{X(\Omega_i)} = \big\|u^{(i)}\big\|_{X(\Omega_1 \cup \Omega_2)}$, $i = 1, 2$, where

$$u^{(1)} = \begin{cases} u(x), & x \in \Omega_1, \\ 0, & x \in \Omega_2, \end{cases} \quad u^{(2)} = \begin{cases} 0, & x \in \Omega_1, \\ u(x), & x \in \Omega_2. \end{cases}$$

The second relation in (3.1) has a similar meaning.

Given two disjoint regular open sets $\Omega_1$ and $\Omega_2$ in $\mathbb{R}^n$, consider the operator $S_{\Omega_1} \dot{+} S_{\Omega_2}$. The domain $D\big(S_{\Omega_1} \dot{+} S_{\Omega_2}\big) = D\big(S_{\Omega_1}\big) \dot{+} D\big(S_{\Omega_2}\big)$ consists of all functions $u$ defined on $\Omega_1 \cup \Omega_2$ for which $u|_{\Omega_i} \in D\big(S_{\Omega_i}\big)$, $i = 1, 2$. Moreover, if $u_1 \in D\big(S_{\Omega_1}\big)$ and $u_2 \in D\big(S_{\Omega_2}\big)$, then the function $u$ defined by (3.2) belongs to $D\big(S_{\Omega_1} \dot{+} S_{\Omega_2}\big)$ and

$$(S_{\Omega_1} \dot{+} S_{\Omega_2})u = \begin{cases} (S_1 u_1)(x), & x \in \Omega_1, \\ (S_2 u_2)(x), & x \in \Omega_2. \end{cases}$$

Also if $u \in D\big(S_{\Omega_1} \dot{+} S_{\Omega_2}\big)$, then $u|_{\Omega_i} \in D\big(S_{\Omega_1} \dot{+} S_{\Omega_2}\big)$, $S_{\Omega_i}(u|_{\Omega_i}) = \big((S_{\Omega_1} \dot{+} S_{\Omega_2})u\big)|_{\Omega_i}$, $i = 1, 2$, and

$$\big\|(S_{\Omega_1} \dot{+} S_{\Omega_2})u\big\|_{Y(\Omega_1 \cup \Omega_2)} = \Big(\big\|S_{\Omega_1}(u|_{\Omega_1})\big\|^2_{Y(\Omega_1)} + \big\|S_{\Omega_2}(u|_{\Omega_2})\big\|^2_{Y(\Omega_2)}\Big)^{1/2}.$$

Given two general regular open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$, we first extend the operators $S_{\Omega_1}$, $S_{\Omega_2}$ to the operators $S_{\Omega_1} \dot{+} S_{\Omega_2 \setminus \overline{\Omega_1}}$, $S_{\Omega_2} \dot{+} S_{\Omega_1 \setminus \overline{\Omega_2}}$ respectively, whose domains lie in the spaces $X\big(\Omega_1 \cup \big(\Omega_2 \setminus \overline{\Omega_1}\,\big)\big)$, $X\big(\Omega_2 \cup \big(\Omega_1 \setminus \overline{\Omega_2}\,\big)\big)$ of functions defined on the open sets $\Omega_1 \cup \big(\Omega_2 \setminus \overline{\Omega_1}\,\big)$ and $\Omega_2 \cup \big(\Omega_1 \setminus \overline{\Omega_2}\,\big)$ which are close but do not coincide. Therefore further extensions are required in order that the domains of the extended operators lie in the same space $X(\Omega_1 \cup \Omega_2)$. To do this we consider the operators $\tilde{S}_{\Omega_1}$, $\tilde{S}_{\Omega_1}$ obtained by extending functions to $\Omega_1 \cup \Omega_2$ by giving them the value 0 in $\Omega_2 \cap \partial\Omega_1$ and in $\Omega_1 \cap \partial\Omega_2$ respectively. Namely,

$$\tilde{S}_{\Omega_1} : D\big(\tilde{S}_{\Omega_1}\big) \subset X(\Omega_1 \cup \Omega_2) \longrightarrow Y(\Omega_1 \cup \Omega_2),$$

where $D\big(\tilde{S}_{\Omega_1}\big)$ is the set of all functions $u$ defined[4] on $\Omega_1 \cup \Omega_2$ such that

$$u|_{\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1}\,)} \in D\big(S_{\Omega_1} \dot{+} S_{\Omega_2 \setminus \overline{\Omega_1}}\big), \quad u(x) = 0 \quad \text{if} \quad x \in \Omega_2 \cap \partial\Omega_1$$

[4] Note that $\Omega_1 \cup \Omega_2 = \big(\Omega_1 \cup \big(\Omega_2 \setminus \overline{\Omega_1}\,\big)\big) \cup (\Omega_2 \cup \partial\Omega_1) = \big(\Omega_2 \cup \big(\Omega_1 \setminus \overline{\Omega_2}\,\big)\big) \cup (\Omega_1 \cup \partial\Omega_2)$.

and for all $u \in D(\tilde{S}_{\Omega_1})$,

$$(\tilde{S}_{\Omega_1} u)|_{\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})} = (S_{\Omega_1} \dotplus S_{\Omega_2 \setminus \overline{\Omega_1}})(u|_{\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})}), \quad (\tilde{S}_{\Omega_1} u)(x) = 0 \quad \text{if} \quad x \in \Omega_2 \cap \partial\Omega_1.$$

$\tilde{S}_{\Omega_2}$ is defined in a similar way.

Since $\left(\overline{\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})}\right)_0 = \left(\overline{\Omega_2 \cup (\Omega_1 \setminus \overline{\Omega_2})}\right)_0 = \Omega_1 \cup \Omega_2$ and the open sets $\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})$, $\Omega_2 \cup (\Omega_1 \setminus \overline{\Omega_2})$ are regular, [5] by property 2)

$$\big\| u|_{\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})} \big\|_{X(\Omega_1) \dotplus X(\Omega_2 \setminus \overline{\Omega_1})} = \|u\|_{X(\Omega_1 \cup \Omega_2)},$$

$$\big\| (\tilde{S}_{\Omega_1} u)|_{\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})} \big\|_{Y(\Omega_1) \dotplus Y(\Omega_2 \setminus \overline{\Omega_1})} = \big\| \tilde{S}_{\Omega_1} u \big\|_{Y(\Omega_1 \cup \Omega_2)}$$

for all $u \in D(\tilde{S}_{\Omega_1})$, and similar equalities hold for the operator $\tilde{S}_{\Omega_2}$.

**Definition 3.1** For regular open sets $\Omega_1, \Omega_2$ in $\mathbb{R}^n$ we define the *gap* between the operators $S_{\Omega_1}$ and $S_{\Omega_2}$ by the equality

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \delta(\tilde{S}_{\Omega_1}, \tilde{S}_{\Omega_2}) \tag{3.3}$$

and the *symmetric gap* by the equality

$$\hat{\delta}(S_{\Omega_1}, S_{\Omega_2}) = \max\{\delta(S_{\Omega_1}, S_{\Omega_2}), \delta(S_{\Omega_2}, S_{\Omega_1})\}, \tag{3.4}$$

where the gap in the right side of (3.3) is the gap between operators acting from the space $X(\Omega_1 \cup \Omega_2)$ to $Y(\Omega_1 \cup \Omega_2)$ (see Definition 2.7).

In more detail

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \sup_{\substack{u \in D(\tilde{S}_{\Omega_1}) \\ \|u\|^2_{X(\Omega_1 \cup \Omega_2)} + \|\tilde{S}_{\Omega_1} u\|^2_{Y(\Omega_1 \cup \Omega_2)} = 1}} \operatorname{dist}(u, D(\tilde{S}_{\Omega_2})),$$

where

$$\operatorname{dist}(u, D(\tilde{S}_{\Omega_2})) = \inf_{v \in D(\tilde{S}_{\Omega_2})} \Big( \|u - v\|^2_{X(\Omega_1 \cup \Omega_2)} + \big\|\tilde{S}_{\Omega_1} u - \tilde{S}_{\Omega_2} v\big\|^2_{Y(\Omega_1 \cup \Omega_2)} \Big)^{1/2}.$$

In applications the operators $S_{\Omega_1}$, $S_{\Omega_1}$ will be of the same nature, say second order elliptic operators with boundary conditions of the same type.

**Remark 3.2** If in the above definition $X(\Omega_i) = Y(\Omega_i)$, $i = 1, 2$, are Hilbert spaces and the operators $S_{\Omega_1}$, $S_{\Omega_2}$, $S_{\Omega_2 \setminus \overline{\Omega_1}}$, $S_{\Omega_1 \setminus \overline{\Omega_2}}$ are self-adjoint (or essentially self-adjoint), then the operators $\tilde{S}_{\Omega_1}$, $\tilde{S}_{\Omega_2}$ are also self-adjoint (essentially self-adjoint), hence the gap between $S_{\Omega_1}$ and $S_{\Omega_2}$ is symmetric

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \delta(S_{\Omega_2}, S_{\Omega_1}) = \hat{\delta}(S_{\Omega_2}, S_{\Omega_1}),$$

see Proposition 2.12 (vii) and Corollary 2.9.

**Example 3.3** Let for each regular open set $\Omega \subset \mathbb{R}$ $X(\Omega) = Y(\Omega) = L^2(\Omega)$, then the assumptions (1)–(3) are satisfied. Moreover, let

$$S_\Omega : D(S_\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

where

$$D(S_\Omega) = \{u \in C^2(\Omega) : u'' \in L^2(\Omega), \quad u|_{\partial\Omega} = 0\}$$

[5] because $(\Omega_1 \cup \Omega_2) \setminus (\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})) = \Omega_2 \cup \partial\Omega_1$, $(\Omega_1 \cup \Omega_2) \setminus (\Omega_2 \cup (\Omega_1 \setminus \overline{\Omega_2})) = \Omega_1 \cup \partial\Omega_2$ and $\operatorname{meas} \partial\Omega_1 = \operatorname{meas} \partial\Omega_2 = 0$

 

and

$$S_\Omega u = -u''$$

for all $u \in D(S_\Omega)$.

Let $S_\varepsilon = S_{]0,1+\varepsilon[}$ for $0 \le \varepsilon \le 1$, then

$$D(S_\varepsilon) = \{u \in C^2(]0, 1+\varepsilon[) : u'' \in L^2(0, 1+\varepsilon),\ u(0) = u(1+\varepsilon) = 0\},$$

where

$$u(0) = \lim_{y\to 0^+} u(y), \quad u(1+\varepsilon) = \lim_{y\to(1+\varepsilon)^-} u(y).$$

By Definition 3.1 and Remark 3.2,

$$\delta(S_0,\, S_\varepsilon) = \delta(S_\varepsilon,\, S_0) = \delta\big(\tilde{S}_\varepsilon,\, \tilde{S}_0\big) = \delta\big(S_\varepsilon,\, \tilde{S}_0\big),$$

where the operator $\tilde{S}_0$ has the domain

$$D\big(\tilde{S}_0\big) = \{u \in C^2(]0,1[\cup]1+\varepsilon[) : u'' \in L^2(0, 1+\varepsilon),\ u(0) = u(1) = u(1+\varepsilon) = 0\}$$

and

$$\tilde{S}_0 u = -u''$$

for all $u \in D\big(\tilde{S}_0\big)$.

Hence

$$\delta(S_\varepsilon, S_0) = \sup_{\substack{v\,\in\, C^2(]0,1+\varepsilon[)\\ v(0)=v(1+\varepsilon)=0\\ \int_0^{1+\varepsilon}(|v|^2+|v''|^2)\,dx=1}} \operatorname{dist}\big(v,\, D\big(\tilde{S}_0\big)\big), \tag{3.5}$$

where

$$\operatorname{dist}\big(v,\, D\big(\tilde{S}_0\big)\big) = \inf_{\substack{u\,\in\, C^2([0,1]\cup[1,1+\varepsilon[)\\ u(0)=u(1)=u(1+\varepsilon)=0}} \left(\int_0^{1+\varepsilon}\big(|v-u|^2 + |v''-u''|^2\big)dx\right)^{1/2}.$$

If $v(1) = 0$, then $\operatorname{dist}\big(v,\, D\big(\tilde{S}_0\big)\big) = 0$ because one can take $u = v$. Let $v(1) \neq 0$. Making the transformation $u \to w = (v-u)/v(1)$, we find for $\varepsilon > 0$

$$\operatorname{dist}\big(v,\, D\big(\tilde{S}_0\big)\big) = k(\varepsilon)|v(1)|, \tag{3.6}$$

where

$$k(\varepsilon) = \inf_{\substack{w\,\in\, C^2(]0,1[\cup]1,1+\varepsilon[)\\ w(0)=w(1+\varepsilon)=0\\ w(1)=1}} \left(\int_0^{1+\varepsilon}\big(|w|^2 + |w''|^2\big)dx\right)^{1/2}. \tag{3.7}$$

Equality (3.6) holds also if $v(1) = 0$, in this case both sides being equal to zero.

Finally, substituting (3.6) into (3.5) we have

$$\delta(S_\varepsilon, S_0) = k(\varepsilon)\cdot \sup_{\substack{v\,\in\, C^2(]0,1+\varepsilon[)\\ v(0)=v(1+\varepsilon)=0\\ \int_0^{1+\varepsilon}(|v|^2+|v''|^2)\,dx=1}} |v(1)|. \tag{3.8}$$

With regards to the factor $k(\varepsilon)$ we have the following estimates.

**Lemma 3.4** *For* $0 < \varepsilon \le 1$

$$\frac{1}{4\sqrt{2}} \le k(\varepsilon) \le \sqrt{\frac{2}{3}}. \tag{3.9}$$

Proof. Let $w \in C^2(]0,1[\cup]1,1+\varepsilon[)$ be such that $w(0) = w(1+\varepsilon) = 0$ and $w(1) = 1$. Then using inequality [6] (3.26) of [2] we get

$$1 = |w(1)| = \left|\int_0^1 w'(x)\,dx\right| \le \int_0^1 |w'(x)|\,dx$$
$$\le 4\int_0^1 (|w(x)| + |w''(x)|)\,dx \le 4\sqrt{2}\left(\int_0^{1+\varepsilon} \left(|w|^2 + |w''|^2\right)dx\right)^{1/2}.$$

Thus the lower bound for $k(\varepsilon)$ in (3.9) follows. The upper bound for $k(\varepsilon)$ is obtained by considering the function $w(x) = x$ for $0 \le x \le 1$ and $w(x) = (1+\varepsilon-x)/\varepsilon$ for $1 \le x \le 1+\varepsilon$ and plugging it into the right-hand side integral of (3.7). □

We have the following estimates about the gap.

**Theorem 3.5**

$$\frac{1}{4\sqrt{2+\pi^2}} \cdot \varepsilon \le \delta(S_\varepsilon, S_0) \le 8\sqrt{\frac{2}{3}} \cdot \varepsilon \tag{3.10}$$

*for all* $0 \le \varepsilon \le 1$.

Proof. Let $0 < \varepsilon \le 1$. Take any $v \in C^2(]0,1+\varepsilon[)$ such that $v(0) = v(1+\varepsilon) = 0$ and $\int_0^{1+\varepsilon} \left(|v|^2 + |v''|^2\right)dx = 1$. By using again inequality (3.26) of [2] (see footnote [6]), we get

$$|v(1)| = \left|-\int_1^{1+\varepsilon} v'(x)\,dx\right| \le \varepsilon\|v'\|_{C([0,1+\varepsilon])}$$
$$\le 4\varepsilon\left(\frac{1}{(1+\varepsilon)^2}\int_0^{1+\varepsilon} |v(x)|\,dx + \int_0^{1+\varepsilon} |v''(x)|\,dx\right)$$
$$\le 4\sqrt{2}\varepsilon\sqrt{1+\varepsilon}\left(\int_0^{1+\varepsilon} \left(|v(x)|^2 + |v''(x)|^2\right)dx\right)^{1/2}$$
$$\le 8\varepsilon.$$

From this estimate and (3.8) we obtain the upper bound in (3.10).

The lower bound of the gap in (3.10) is obtained by plugging in (3.8) the eigenfunction

$$v_\varepsilon(x) = c_\varepsilon \sin\frac{\pi x}{1+\varepsilon}$$

---

[6] For conveninence we formulate this inequality. Let $a < b$ be real numbers and $u$ a differentiable function on $[a,b]$ with derivative $u'$ absolutely continuous on $[a,b]$. Then for all $x \in [a,b]$

$$|u'(x)| \le 4\left(\frac{1}{(b-a)^2}\int_a^b |u(y)|dy + \int_a^b |u''(y)|dy\right).$$

 

corresponding to the smallest nonzero eigenvalue $\left(\frac{\pi}{1+\varepsilon}\right)^2$ of the operator $S_\varepsilon$, with the normalization constant $c_\varepsilon > 0$ such that

$$\int_0^{1+\varepsilon} \left(|v_\varepsilon(x)|^2 + \left|v_\varepsilon''(x)\right|^2\right) dx = 1,$$

and by taking into account the lower bound of $k(\varepsilon)$ in (3.9). After simple calculations we obtain

$$c_\varepsilon = \left[1/2\left(1+\varepsilon+\frac{\pi^2}{1+\varepsilon}\right)\right]^{-1/2}.$$

Thus

$$\delta(S_\varepsilon, S_0) \geq \frac{1}{4}\left(1+\varepsilon+\frac{\pi^2}{1+\varepsilon}\right)^{-1/2} \sin\frac{\pi}{1+\varepsilon},$$

and from this, using the fact that $\sin\left(\pi/(1+\varepsilon)\right) \geq \varepsilon$, we get the first inequality in (3.10). □

Now let us consider the analogues of these operators in multiple dimensions.

**Example 3.6** Let $X(\Omega) = Y(\Omega) = L^2(\Omega)$ for each regular open set $\Omega \subset \mathbb{R}^n$. Then the assumptions (1)–(3) are satisfied. Moreover, let $S_\Omega = -\Delta_{D,\Omega}$ (see Section 2.4).

If $\Omega_1, \Omega_2$ are regular bounded open sets, we have

$$\delta\big(-\Delta_{D,\Omega_2}, -\Delta_{D,\Omega_1}\big) = \sup_{\substack{u \in H_0^1(\Omega_1) \cap H^\Delta(\Omega_1) \\ \cap H_0^1(\Omega_2 \setminus \overline{\Omega_1}) \cap H^\Delta(\Omega_2 \setminus \overline{\Omega_1}) \\ \|u\|^2_{H^\Delta(\Omega_1)} + \|u\|^2_{H^\Delta(\Omega_2 \setminus \overline{\Omega_1})} = 1}} dist\big(u,\, D\big(\tilde{S}_{\Omega_2}\big)\big),$$

where

$$dist\big(u,\, D\big(\tilde{S}_{\Omega_2}\big)\big) = \inf_{\substack{v \in H_0^1(\Omega_2) \cap H^\Delta(\Omega_2) \\ \cap H_0^1(\Omega_1 \setminus \overline{\Omega_2}) \cap H^\Delta(\Omega_1 \setminus \overline{\Omega_2})}} \mathcal{H}(u,v)$$

with

$$\mathcal{H}(u,v) = \left(\|u-v\|^2_{L^2(\Omega_1\cup\Omega_2)} + \|\Delta u - \Delta v\|^2_{L^2(\Omega_1\cup\Omega_2)}\right)^{1/2}$$

for all $u \in H^\Delta(\Omega_1) \cap H^\Delta\big(\Omega_2 \setminus \overline{\Omega_1}\big)$ and $v \in H^\Delta(\Omega_2) \cap H^\Delta\big(\Omega_1 \setminus \overline{\Omega_2}\big)$ (note that functions $\Delta u$ and $\Delta v$ are defined on $\Omega_1 \cup \Omega_2$ unless subsets of measure zero–for example, $\Delta u(x)$ is not defined for $x \in \Omega_2 \cap \partial\Omega_1$ and meas $(\Omega_2 \cap \partial\Omega_1) = 0$–and therefore, $\mathcal{H}(u,v)$ is well-defined).

If $\overline{\Omega_2} \subset \Omega_1$ and if $\Omega_1, \Omega_2$ have $C^{1,r}$ boundaries, $1/2 < r \leq 1$, and $n = 1, 2, 3$, by Theorem 2.14, we have

$$\delta\big(-\Delta_{D,\Omega_1}, -\Delta_{D,\Omega_2}\big) = \sup_{\substack{u \in C^2(\Omega_1) \\ u|_{\partial\Omega_1} = 0 \\ \mathcal{K}(u) = 1}} \inf_{\substack{v \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2}) \\ \Delta v \in L^2(\Omega_2) \cap L^2(\Omega_1 \setminus \overline{\Omega_2}) \\ v|_{\partial\Omega_1} = 0,\ v|_{\partial\Omega_2} = 0}} \mathcal{K}(u-v) \tag{3.11}$$

and

$$\delta\big(-\Delta_{D,\Omega_2}, -\Delta_{D,\Omega_1}\big) = \sup_{\substack{v \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2}) \\ v|_{\partial\Omega_1} = 0,\ v|_{\partial\Omega_2} = 0 \\ \mathcal{K}(v) = 1}} \inf_{\substack{u \in C^2(\Omega_1) \\ \Delta u \in L^2(\Omega_1) \\ u|_{\partial\Omega_1} = 0,}} \mathcal{K}(v-u),$$

where

$$\mathcal{K}(w) = \left(\|w\|^2_{L^2(\Omega_1)} + \|\Delta w\|^2_{L^2(\Omega_1)}\right)^{1/2} \tag{3.12}$$

for every $w \in C^2(\Omega_2) \cap C^2\big(\Omega_1 \setminus \overline{\Omega_2}\big)$.

If we make the transformation $v \to w = u - v$ in (3.11), we find

$$\delta\big(-\Delta_{D,\Omega_1}, -\Delta_{D,\Omega_2}\big) = \sup_{\substack{u \in C^2(\Omega_1)\\ u|_{\partial\Omega_1} = 0\\ \mathcal{K}(u) = 1}} \inf_{\substack{w \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2})\\ w|_{\partial\Omega_2} = u|_{\partial\Omega_2},\ w|_{\partial\Omega_1} = 0}} \mathcal{K}(w). \tag{3.13}$$

## 4 Estimates of the gap between operators defined on different open sets

### 4.1 The case of the Dirichlet Laplacian

In the present section we obtain estimates of the gap between some of the operators considered in Example 3.6. To estimate $\delta\big(-\Delta_{D,\Omega_1}, -\Delta_{D,\Omega_2}\big)$ we need to prove the following Lemma 4.2. For this purpose it is necessary to know under what conditions the classical Dirichlet problem associated to the Laplace equation with continuous boundary data is solvable. We recall the following fact: the Dirichlet problem associated to the Laplace equation has a (unique) solution in $C^2(\Omega) \cap C\big(\overline{\Omega}\big)$ for arbitrary continuous boundary values if and only if the boundary points are *regular* (with respect to the Laplacian). For this fact and the definition of a regular boundary point of an open set see [6], pages 25–26 and, in particular, Theorem 3.14. For an open set to have regular boundary points sufficient general conditions can be given in terms of local geometrical properties (see again [6]). We limit ourselves to a simple one of such conditions – *the exterior cone condition*: an open set $\Omega \subset \mathbb{R}^n$ is said to satisfy *an exterior cone condition* if for all $x_0 \in \partial\Omega$ there exists a finite right circular cone $K$, with vertex $x_0$, satisfying $\overline{K} \cap \Omega = \{x_0\}$. Furthermore, an open set $\Omega$ is said to satisfy *an interior cone condition* if the open set $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies an exterior cone condition.

We recall that an open set with a Lipschitz boundary satisfies both an exterior and an interior cone condition.

Let $\Omega \subset \mathbb{R}^n$ and $\varepsilon > 0$. We define

$$\Omega_\varepsilon = \{x \in \Omega \ :\ \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) > \varepsilon\}.$$

With these notations the main result is the following:

**Theorem 4.1** *Let $n = 2, 3$ and $0 < \gamma < 1$ if $n = 2$, $\gamma = 1/2$ if $n = 3$. Moreover, let $\Omega_1 \subset \mathbb{R}^n$ be a bounded open set with a $C^{1,1}$ boundary. Then there exists $M > 0$ such that*

$$\delta\big(-\Delta_{D,\Omega_1}, -\Delta_{D,\Omega_2}\big) \le M\varepsilon^\gamma, \tag{4.1}$$

*for all $\varepsilon > 0$ and for all open sets $\Omega_2$ with $C^{1,r}$, $1/2 < r \le 1$, boundaries such that $(\Omega_1)_\varepsilon \subset \overline{\Omega_2} \subset \Omega_1$.*

For the proof of this theorem we need a pair of lemmas. In the following lemma and in the sequel $\mathcal{K}$ is as in (3.12).

**Lemma 4.2** *Let $n \ge 2$, $\Omega_1$, $\Omega_2$ be bounded open sets in $\mathbb{R}^n$ such that $\overline{\Omega_2} \subset \Omega_1$ and all boundary points of $\Omega_2$ and $\Omega_1 \setminus \overline{\Omega_2}$ are regular with respect to the Laplacian. Assume that $u \in C^2(\Omega_1)$, $\Delta u \in L^2(\Omega_1)$ and $u|_{\partial\Omega_1} = 0$. Then*

$$I(u) = \inf_{\substack{w \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2})\\ w|_{\partial\Omega_2} = u|_{\partial\Omega_2},\ w|_{\partial\Omega_1} = 0}} \mathcal{K}(w) \le (\mathit{meas}\ \Omega_1)^{1/2} \|u|_{\partial\Omega_2}\|_{C(\partial\Omega_2)}. \tag{4.2}$$

P r o o f. Let $w' \in C^2(\Omega_2)$ be such that

$$\begin{cases} \Delta w' = 0 & \text{in} \quad \Omega_2,\\ w'|_{\partial\Omega_2} = u|_{\partial\Omega_2} \end{cases}$$

and $w'' \in C^2(\Omega_1 \setminus \overline{\Omega_2})$ be such that

$$\begin{cases} \Delta w'' = 0 & \text{in} \quad \Omega_1 \setminus \overline{\Omega_2},\\ w''|_{\partial\Omega_2} = u|_{\partial\Omega_2},\\ w''|_{\partial\Omega_1} = 0. \end{cases}$$

Such functions $w'$ and $w''$ exist because all the boundary points of $\Omega_2$ and $\Omega_1 \setminus \overline{\Omega_2}$ are regular with respect to the Laplacian.

We put

$$w_0 = \begin{cases} w' & \text{in} \quad \Omega_2, \\ w'' & \text{in} \quad \overline{\Omega_1} \setminus \Omega_2. \end{cases}$$

Therefore

$$\begin{aligned} I(u) &\le \mathcal{K}(w_0) \\ &= \left( \int_{\Omega_1} |w_0|^2 dx \right)^{1/2} \\ &= \left( \int_{\Omega_2} |w'|^2 dx + \int_{\Omega_1 \setminus \Omega_2} |w''|^2 dx \right)^{1/2}. \end{aligned}$$

By the maximum principle (see, for instance, [6, Corollary 3.2, page 33])

$$\begin{aligned} I(u) &\le \left( \text{meas } \Omega_2 \|w'\|^2_{C(\Omega_2)} + \text{meas } (\Omega_1 \setminus \Omega_2) \|w''\|^2_{C(\Omega_1 \setminus \overline{\Omega_2})} \right)^{1/2} \\ &= \left( \text{meas } \Omega_2 \|u|_{\partial\Omega_2}\|^2_{C(\partial\Omega_2)} + \text{meas } (\Omega_1 \setminus \Omega_2) \|u|_{\partial\Omega_2}\|^2_{C(\partial\Omega_2)} \right)^{1/2} \\ &= (\text{meas } \Omega_1)^{1/2} \|u|_{\partial\Omega_2}\|_{C(\partial\Omega_2)}. \end{aligned}$$

□

From this lemma and (3.13), we infer immediately the following corollary.

**Corollary 4.3** *Let $n = 2, 3$, $\Omega_1 \subset \mathbb{R}^n$ be a bounded open set with a $C^{1,r}$ boundary for some $1/2 < r \le 1$ and let $\Omega_2$ be a bounded open set such that $\overline{\Omega_2} \subset \Omega_1$ and all boundary points of $\Omega_2$ and $\Omega_1 \setminus \overline{\Omega_2}$ are regular with respect to the Laplacian. Then*

$$\delta\left(-\Delta_{D,\Omega_1}, -\Delta_{D,\Omega_2}\right) \le (\textit{meas } \Omega_1)^{1/2} \sup_{\substack{u \in C^2(\Omega_1) \\ u|_{\partial\Omega_1} = 0 \\ \|u\|^2_{L^2(\Omega_1)} + \\ \|\Delta u\|^2_{L^2(\Omega_1)} = 1}} \|u|_{\partial\Omega_2}\|_{C(\partial\Omega_2)}. \tag{4.3}$$

To proceed further with the estimation of the gap we need to investigate under what conditions on $\Omega$ the inequality

$$\sup_{x \in \Omega} (\text{dist}(x, \partial\Omega))^{-\gamma} |u(x)| \le M_0 \left( \int_{\Omega} \left(|u|^2 + |\Delta u|^2\right) dx \right)^{1/2} \tag{4.4}$$

holds, where $u \in C^2(\Omega)$, $u|_{\partial\Omega} = 0$ and $0 < \gamma \le 1$, $M_0 > 0$ are fixed constants independent of the function $u$. With this regard we have the following result.

**Lemma 4.4** *Let $n = 2, 3$ and $0 < \gamma < 1$ if $n = 2$, $\gamma = 1/2$ if $n = 3$. Moreover, let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a $C^{1,1}$ boundary. Then there exists $M_0 > 0$ such that estimate* (4.4) *holds for any $u \in C^2(\Omega)$ such that $\Delta u \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$.*

Proof. Since $\Omega$ is an open set with a $C^{1,1}$ boundary, it is a known fact (see, for example, [9, Remark 6.1, page 70]) that there exists a constant $M_1 > 0$ independent of $u$ such that

$$\|u\|_{H^2(\Omega)} = \left( \int_\Omega \left( |u|^2 + |\nabla^2 u|^2 \right) dx \right)^{1/2} \le M_1 \left( \int_\Omega \left( |u|^2 + |\Delta u|^2 \right) dx \right)^{1/2}.$$

Moreover, also the Sobolev's embedding $H^2(\Omega) \subset C^\gamma(\overline{\Omega})$ holds[7]. Thus, we have

$$\sup_{x\in\Omega} (\mathrm{dist}(x, \partial\Omega))^{-\gamma} |u(x)| \le \sup_{x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\gamma} \le \|u\|_{C^\gamma(\overline{\Omega})} \le M_2 \|u\|_{H^2(\Omega)},$$

for some $M_2 > 0$ independent of $u$, which together with the preceding inequality implies the desired estimate (4.4) with $M_0 = M_1 M_2$. □

**Remark 4.5** Inequality (4.4) does not hold if $n = 4$ and $0 < \gamma < 1/2$. Indeed, consider the function

$$u(x) = (-\ln|x|)^\gamma - (\ln 2)^\gamma \quad \text{in} \quad \Omega = \left\{ x \in \mathbb{R}^4 : |x| < 1/2 \right\}.$$

Then

$$|\Delta u| \le c \cdot (-\ln|x|)^{\gamma-1} |x|^{-2} \in L^2(\Omega),$$

for some constant $c > 0$, so that the right side in (4.4) is finite, while the left side is $+\infty$.

*Proof of Theorem 4.1.* This statement is a consequence of Corollary 4.3 and Lemma 4.4. Indeed, let $u \in C^2(\Omega_1)$ be such that $\mathcal{K}(u) = 1$, $u|_{\partial\Omega_1} = 0$ and $z \in \partial\Omega_2$. Let also $y \in \partial\Omega_1$ be such that $\mathrm{dist}(z, \partial\Omega_1) = |z-y| > 0$. Then by inequality (4.4) we have

$$\begin{aligned} |u(z)| &= |z-y|^\gamma \frac{|u(z)|}{|z-y|^\gamma} \\ &\le \varepsilon^\gamma \sup_{x\in\Omega_1} \frac{|u(x)|}{\mathrm{dist}(x, \partial\Omega_1)^\gamma} \\ &\le M_0 \varepsilon^\gamma \mathcal{K}(u) \\ &= M_0 \varepsilon^\gamma. \end{aligned}$$

Hence, by (4.3), we obtain (4.1) with $M = M_0 \left(\mathrm{meas}\, \Omega_1\right)^{1/2}$. □

### 4.2 Extension of gap estimates to general elliptic operators

The estimates of Section 4.1 about the gap between the Dirichlet Laplacians defined on different open sets extend straightforwardly to more general elliptic operators. Actually, in obtaining the aforementioned results, the only ingredients that we have used were: the solvability of the corresponding homogeneous equation (Laplace equation) for continuous boundary data, the maximum principle, an $H^2$-global regularity estimate and the Sobolev embedding theorem. Well, these facts hold for a vast class of elliptic operators. For the sake of clarity, in the sequel we shall summarily repeat the procedure.

[7] As usual, $C^\gamma(\overline{\Omega})$ is the set of all functions $u$ continuous on $\overline{\Omega}$ for which

$$\|u\|_{C^\gamma(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |u(x)| + \sup_{x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\gamma} < \infty.$$

 

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $(a_{ij})_{i,j=1,\dots,n}$ be a Hermitian matrix of Lipschitz continuous functions and suppose that there exists a positive constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \geq \theta \sum_{i=1}^{n} |\xi_i|^2, \tag{4.5}$$

for all $x \in \Omega$ and for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$. We shall consider uniformly elliptic second order differential operators in divergence form, that is, operators that derive from differential expressions of the type

$$Su = -\sum_{i,j=1}^{n} \big(a_{ij}(x)u_{x_i}\big)_{x_j} + c(x)u, \tag{4.6}$$

whose second order coefficients $(a_{i\,j})_{i,\,j=1,\dots,n}$ satisfy the aforementioned conditions and, in addition, $c \in C\big(\overline{\Omega}\big)$ and for some positive constant $c$

$$c(x) \geq c \tag{4.7}$$

for every $x \in \Omega$. With the operator $S$ we shall consider the homogenous Dirichlet boundary conditions. The *maximal classical Dirichlet realization* in $L^2(\Omega)$ of the formal differential expression $S$ in (4.6), which will be denoted by $S_\Omega$ (to emphasize also its dependence on the open set $\Omega$), is a linear operator, which is defined in the following way:

$$D(S_\Omega) = \{u \in C^2(\Omega) \,:\, u|_{\partial\Omega} = 0 \text{ and } Su \in L^2(\Omega)\} \tag{4.8}$$

and

$$S_\Omega u = Su \quad \text{for every} \quad u \in D(S_\Omega). \tag{4.9}$$

Clearly $C_c^\infty(\Omega) \subset D(S_\Omega)$, hence the operator $S_\Omega$ is densely defined in $L^2(\Omega)$.

It is easily seen that $S_\Omega$ is a closable operator; the domain $D(\overline{S_\Omega})$ of its closure $\overline{S_\Omega}$ in $L^2(\Omega)$ is the completion of $D(S_\Omega)$ with respect to the norm

$$u \longrightarrow \|u\|_{H^S(\Omega)} = \big(\|u\|^2_{L^2(\Omega)} + \|Su\|^2_{L^2(\Omega)}\big)^{1/2}, \quad u \in C^2(\Omega). \tag{4.10}$$

We find it useful to introduce the space $H^S(\Omega)$ which is, by definition, the completion of $\{u \in C^2(\Omega) : \|u\|_{H^S(\Omega)} < \infty\}$ with respect to norm (4.10).

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be bounded open sets such that $\overline{\Omega_2} \subset \Omega_1$. Let $S_{\Omega_i}$, $i = 1, 2$, be linear second order elliptic operators associated to each of the open sets $\Omega_i$ as introduced above by (4.8) and (4.9) (the coefficients $a_{ij}$, $i, j = 1, \dots, n$, $b$ are defined on the bigger of these sets $\Omega_1$). By Definition 3.1, the gap between these operators is

$$\delta\big(S_{\Omega_1}, S_{\Omega_2}\big) = \sup_{\substack{u \in C^2(\Omega_1) \\ \|u\|_{H^S(\Omega_1)} = 1}} I_S(u) \tag{4.11}$$

where

$$I_S(u) = \inf_{\substack{w \in C^2(\Omega_2) \cap C^2(\Omega_1 \setminus \overline{\Omega_2}) \\ w|_{\partial\Omega_2} = u|_{\partial\Omega_2},\ w|_{\partial\Omega_1} = 0}} \big(\|w\|^2_{L^2(\Omega_1)} + \|Sw\|^2_{L^2(\Omega_1)}\big)^{1/2}.$$

From now on we suppose that $\Omega_2,\ \Omega_1 \setminus \overline{\Omega_2}$ satisfy an exterior cone condition at the boundary points (more generally, we can suppose that all their boundary points are regular). Observe that $\Omega_1 \setminus \overline{\Omega_2}$ satisfies an exterior cone condition if and only if $\Omega_1$ satisfies an exterior cone condition and $\Omega_2$ satisfies an interior cone condition. Proceeding in the same way as in Lemma 4.2, using a theorem on the classical solvability of the Dirichlet problem with continuous boundary data associated to uniformly second order elliptic equations (see [6, Theorem 6.13, p. 106] and the following comments), and the maximum principle (see [6, Corollary 3.2, p. 33]), we find that for all $u \in D\big(S_{\Omega_1}\big)$

$$\delta\big(S_{\Omega_1},\, S_{\Omega_2}\big) \leq (\text{meas}\,\Omega_1)^{1/2}\,\|u\|_{C(\partial\Omega_2)}. \tag{4.12}$$

Now let the dimension $n = 2, 3$. Let $\gamma = 1/2$ if $n = 3$ and $0 < \gamma < 1$ if $n = 2$. In addition to an exterior cone condition assume that the bounded open set $\Omega_1$ is such that the following global regularity estimate holds for the solutions of second order elliptic equations that vanish at the boundary, that is, there exists a positive constant $M_0$ depending only on $\Omega_1$ such that

$$\|u\|_{H^2(\Omega_1)} \leq M_0\|u\|_{H^S(\Omega_1)} \tag{4.13}$$

for all $u \in C^2(\Omega_1)$ with $Su \in L^2(\Omega_1)$ and $u|_{\partial\Omega_1} = 0$. For this to hold, it is sufficient that $\Omega_1$ be an open set with a $C^{1,1}$ boundary or an open set which is convex or a polyhedron. For more conditions on $\Omega_1$ that guarantee the aforementioned regularity estimate see [9, Remark 6.1, p. 70] (for a convex set (4.13) was first proved by Kadlec in [7]). In addition, assume that $\Omega_1$ is such that the Sobolev embedding

$$H^2(\Omega_1) \subset C^\gamma\left(\overline{\Omega_1}\right)$$

holds. It is certainly so if $\Omega$ is a convex set or a polyhedron or if $\Omega_1$ has a $C^{1,1}$ boundary. In the same fashion as in the proof of Lemma 4.4, using this global regularity estimate (4.13) and the above Sobolev embedding, we obtain

$$\sup_{x\in\Omega_1} \operatorname{dist}(x, \partial\Omega_1)^{-\gamma}|u(x)| \leq M_1\|u\|_{H^S(\Omega_1)}, \tag{4.14}$$

where the constant $M_1 > 0$ is independent of $u$.

Finally, using this last estimate as in the proof of Theorem 4.1 we obtain the analogous result.

**Theorem 4.6** *Let $n = 2, 3$ and $\gamma = 1/2$ if $n = 3$, $0 < \gamma < 1$ if $n = 2$. Let $\Omega_1$ be a bounded open set with a $C^{1,1}$ boundary or an open convex set or an open polyhedron, then there exists a constant $M > 0$ such that*

$$\delta\big(S_{\Omega_1},\, S_{\Omega_2}\big) \leq M\varepsilon^\gamma \tag{4.15}$$

*for all $\varepsilon > 0$ and for all open sets $\Omega_2$ that satisfy an interior and an exterior cone condition and are such that $(\Omega_1)_\varepsilon \subset \Omega_2 \subset \overline{\Omega_2} \subset \Omega_1$.*

A important case is when the operators $S_{\Omega_1}$,, $S_{\Omega_2}$, $S_{\Omega_1\setminus\overline{\Omega_2}}$ are essentially self-adjoint; this is the case when the open sets $\Omega_1$, $\Omega_2$ have a $C^{1,1}$ boundary (see [17]). In this situation, as we noted in Remark 3.2, the gap between these operators is symmetric. In a forthcoming paper, we shall use the previous estimates of the gap to obtain stability estimates for the eigenfunctions of the closures of these operators.

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