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Spectral stability estimates for the eigenfunctions of second order elliptic operators

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Stability of the eigenfunctions of nonnegative selfadjoint second-order linear elliptic operators subject to homogeneous Dirichlet boundary data under domain perturbation is investigated. Let $\Omega, \Omega' \subset \mathbb{R}^n$ be bounded open sets. The main result gives estimates for the variation of the eigenfunctions under perturbations Ω' of Ω such that $\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x, \mathbb{R}^n \backslash \Omega) > \varepsilon\} \subset \Omega' \subset \overline{\Omega'} \subset \Omega$ in terms of powers of ε, where the parameter $\varepsilon > 0$ is sufficiently small. The estimates obtained here hold under some regularity assumptions on Ω, Ω'. They are obtained by using the notion of a gap between linear operators, which has been recently extended by the authors to differential operators defined on different open sets.

1 Introduction

Let $(a_{ij})_{i,j=1,\dots,n}$ be a Hermitian matrix of Lipschitz continuous complex-valued functions defined on $\mathbb{R}^n$ and suppose that there exists a positive constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\overline{\xi_j} \geq \theta \sum_{i=1}^{n} |\xi_i|^2, \tag{1.1}$$

for all $x \in \mathbb{R}^n$ and for all $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{C}^n$. Let also $b \in C(\mathbb{R}^n)$ be a real-valued function such that for some positive constant c

$$b(x) \geq c$$

for every $x \in \mathbb{R}^n$.

For every bounded open set $\Omega \subset \mathbb{R}^n$ we consider the eigenvalue problem

$$\begin{cases} Su = \lambda u & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where S is the uniformly elliptic second order linear differential operator in divergence form

$$Su = -\sum_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + b(x)u, \quad x \in \Omega. \tag{1.3}$$

The problem has a standard *weak formulation* that goes as follows: consider the nonnegative sesquilinear form

$$Q : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{C} \tag{1.4}$$

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in $L^2(\Omega)$ defined by

$$Q(u,v) = \int_\Omega \left(\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i}(x) \overline{v_{x_j}(x)} + b(x) u(x) \overline{v(x)} \right) dx \tag{1.5}$$

for all $u, v \in H_0^1(\Omega)$. It is a well-known result (see, e.g., [7, Chapter 6]) that there exists a nonnegative selfadjoint operator S_Ω such that $u \in D(S_\Omega)$ if and only if $u \in H_0^1(\Omega)$ and there exists an $f \in L^2(\Omega)$ such that

$$Q(u,\, v) = (f,\, v)_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$; in such a case $S_\Omega u = f$. In the weak formulation problem (1.2) is the problem of finding the eigenvalues and eigenfunctions of the operator S_Ω.

The operator S_Ω has compact resolvent, hence its spectrum is discrete, that is, it consists of isolated eigenvalues of finite multiplicity (see again [7]). These eigenvalues are all strictly positive and if arranged in a sequence in nondecreasing order repeating them as many times as their multiplicity,

$$\lambda_1[\Omega] \le \lambda_2[\Omega] \le \cdots \le \lambda_k[\Omega] \le \cdots, \tag{1.6}$$

then $\lim_{k\to\infty} \lambda_k[\Omega] = \infty$. Moreover, there exists an orthonormal basis of eigenfunctions $\{\varphi_k[\Omega]\}_{k\in\mathbb{N}}$ in $L^2(\Omega)$. Here, for each $k \in \mathbb{N}$, $\varphi_k[\Omega]$ is an eigenfunction for the eigenvalue $\lambda_k[\Omega]$. Given $k, m \in \mathbb{N}$ such that

$$\lambda_{k-1}[\Omega] < \lambda_k[\Omega] \le \cdots \le \lambda_{k+m-1}[\Omega] < \lambda_{k+m}[\Omega], \tag{1.7}$$

(the first inequality on the left is not present if $k = 1$), we set

$$N_{k,\,m}[\Omega] = \operatorname{span}\{\varphi_k[\Omega], \ldots, \varphi_{k+m-1}[\Omega]\}. \tag{1.8}$$

Consider a fixed bounded open set Ω. The main result of the paper is Theorem 3.3 which estimates the gap between $N_{k,\,m}[\Omega]$, $N_{k,\,m}[\Omega']$ (seen as subspaces of $L^2(\Omega_1 \cup \Omega_2)$) in terms of the gap between S_Ω and $S_{\Omega'}$ for perturbations Ω' of Ω sufficiently "close" to Ω.

Let $\varepsilon > 0$ be sufficiently small and let the open set Ω' be a perturbation of Ω such that

$$\Omega_\varepsilon \subset \Omega' \subset \overline{\Omega'} \subset \Omega,$$

where

$$\Omega_\varepsilon = \{x \in \Omega : \operatorname{dist}(x,\, \mathbb{R}^n \setminus \Omega) < \varepsilon\}. \tag{1.9}$$

(Thus, ε is a parameter that controls the magnitude of a perturbation Ω' of the open set Ω.) Then Theorem 3.5 gives an estimate of the gap between the two subspaces $N_{k,\,m}[\Omega]$ and $N_{k,\,m}[\Omega']$ (here $N_{k,\,m}[\Omega']$ is seen as a subspace of $L^2(\Omega)$ by extending its elements up to Ω by zero) in terms of powers of ε in space dimension $n \le 3$. This in turn leads to some estimates of the variation of the eigenfunctions $\|\varphi[\Omega'] - \varphi[\Omega]\|_{L^2(\Omega)}$ in terms of ε in Theorem 3.6.

The estimates of this paper are obtained by using the notions of a gap between linear subspaces and between linear operators, which was first introduced by Kreĭn and coworkers in the 1940s (e.g., see [11], [12]). For the convenience of the reader the definition and some of the properties of the gap have been reported in Section 2. In order to adapt it to domain perturbation problems, an extension of the notion of a gap between operators defined on different open sets is needed; a possible definition of such a gap has been suggested in an our recent article [4]. Moreover, in that same article the gap between operators of the kind that we consider here is estimated in terms of ε. For commodity, the said definition and estimate are reported here in Section 3. Finally, this estimate of the gap is used in Section 3 to obtain estimates for the variation of the eigenfunctions (see Theorems 3.5 and 3.6).

The problem of spectral stability is one of the important problems in the theory of partial differential operators, which apart from theoretical interest, is also useful in applications, first of all to numerical methods related to computing eigenvalues and eigenfunctions.

The problem of estimating the variation of eigenvalues upon a domain perturbation has been treated by several authors, see Burenkov, Lamberti and Lanza de Cristoforis [5] for extensive references on this subject. However, much much less attention has been devoted to the problem of finding explicit estimates for the variation of the eigenfunctions. In [1] and [15], [16] only the first nontrivial eigenfunctions of the Dirichlet Laplacian and Neuman Laplacian, respectively, are considered. In the recent paper [2] the case of domain perturbations obtained by suitable diffeomorphisms is considered (see also [13], [14]).

We also note the paper [19] in which sharp stability estimates for the solutions of the Dirichlet problem for the equation $Su = f$ on Ω and Ω' are obtained.

2 Notation and preliminary results

The notion of a gap from-to and between two subspaces of a normed space or two linear operators acting in normed spaces and the relative properties needed here are recalled. We reccommend [9, Chapter 4] for all the facts concerning the notion of a gap between subspaces and between linear operators stated here without proof.

2.1 Gap between subspaces

Definition 2.1 Let $\mathcal{M}$ and $\mathcal{N}$ be linear subspaces of a normed space $\mathcal{Z}$. The *gap from $\mathcal{M}$ to $\mathcal{N}$* is defined by the following formula:

$$\delta(\mathcal{M}, \mathcal{N}) = \sup_{\substack{u \in \mathcal{M} \\ \|u\| = 1}} \operatorname{dist}(u, \mathcal{N}), \tag{2.1}$$

where $\operatorname{dist}(u, \mathcal{N}) = \inf_{v \in N} \|u - v\|$ is the distance of the vector u to the subspace N. (The equality $\|u\| = 1$ can be replaced by the inequality $\|u\| \leq 1$ without changing the result.)

One defines the *gap between $\mathcal{M}$ and $\mathcal{N}$* by

$$\hat{\delta}(\mathcal{M}, \mathcal{N}) = \max\{\delta(\mathcal{M}, \mathcal{N}),\ \delta(\mathcal{N}, \mathcal{M})\}. \tag{2.2}$$

The supremum in (2.1) has no meaning if $\mathcal{M} = \{0\}$; in this case one defines $\delta(\{0\}, \mathcal{N}) = 0$ for any $\mathcal{N}$. On the other hand $\delta(\mathcal{M}, \{0\}) = 1$ if $\mathcal{M} \neq \{0\}$ as is seen by the definition.

Theorem 2.2 *If $\mathcal{M}$ and $\mathcal{N}$ are linear subspaces of a Hilbert space $\mathcal{Z}$, then*

$$\delta(\mathcal{M}, \mathcal{N}) = \|(1 - Q)P\|,$$
$$\hat{\delta}(\mathcal{M}, \mathcal{N}) = \|P - Q\|,$$

where P, Q are the orthogonal projectors onto the closures of $\mathcal{M}$ and $\mathcal{N}$ respectively.

Thus the gap between closed linear subspaces is a metric on closed linear subspaces of a Hilbert space. For finite dimensional subspaces of a Hilbert space one can introduce the notion of principal or canonical angles between the said subspaces (see, e.g., [10]) and the gap between these subspaces in the sine of the greatest of the principal angles.

We need also the following property (a proof of which can be found also in [4].

Theorem 2.3 *Let $\mathcal{M}$ and $\mathcal{N}$ be linear subspaces of a normed space $\mathcal{Z}$. $\delta(\mathcal{M}, \mathcal{N}) < 1$ implies* $\dim\mathcal{M} \leq \dim\mathcal{N}$. $\hat{\delta}(\mathcal{M}, \mathcal{N}) < 1$ *implies* $\dim \mathcal{M} = \dim \mathcal{N}$.

2.2 Gap between operators

Let $\mathcal{X} \dotplus \mathcal{Y}$ denote the direct sum of two normed spaces $\mathcal{X}$, $\mathcal{Y}$, that is, the cartesian product of $\mathcal{X}$ and $\mathcal{Y}$ made into a normed space by the usual definition of addition, multiplication by scalars and norm given by

$$\|(u, v)\|_{\mathcal{X} \times \mathcal{Y}} = \left(\|u\|_{\mathcal{X}}^2 + \|v\|_{\mathcal{Y}}^2\right)^{1/2}$$

for all $u \in \mathcal{X}$, $v \in \mathcal{Y}$. If $\mathcal{X}$ and $\mathcal{Y}$ are inner product spaces with inner products $(\cdot, \cdot)_{\mathcal{X}}$, $(\cdot, \cdot)_{\mathcal{Y}}$ respectively, then $\mathcal{X} \dotplus \mathcal{Y}$, denoted this time more commonly by $\mathcal{X} \oplus \mathcal{Y}$, is also an inner product space with the inner product given by

$$((u, v), (u', v'))_{\mathcal{X} \oplus \mathcal{Y}} = (u, u')_{\mathcal{X}} + (v, v')_{\mathcal{Y}}$$

for all $u, u' \in \mathcal{X}$, $v, v' \in \mathcal{Y}$. Thus, if $\mathcal{X}$, $\mathcal{Y}$ are Banach (Hilbert) spaces, then $\mathcal{X} \dotplus \mathcal{Y}$ ($\mathcal{X} \oplus \mathcal{Y}$) is also a Banach (Hilbert) space. As usual, the subscripts in the notation for norms and inner products will be dropped when there is no danger of ambiguity.

Let

$$T : D(T) \subset \mathcal{X} \longrightarrow \mathcal{Y}$$

be a linear operator, where the linear subspace $D(T)$ of $\mathcal{X}$ is the domain of the operator T. The kernel of the operator T is denoted by $N(T)$ and its range by $R(T)$. The linear subspace

$$G(T) = \{(u, Tu) : u \in D(T)\},$$

of $\mathcal{X} \dotplus \mathcal{Y}$ is the graph of the operator T. We denote by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ ($\mathcal{B}(\mathcal{X})$, if $\mathcal{X} = \mathcal{Y}$) the normed space (normed algebra, if $\mathcal{X} = \mathcal{Y}$) of all bounded linear operators T acting from $\mathcal{X}$ to $\mathcal{Y}$ with domain $D(T) = \mathcal{X}$.

Definition 2.4 Let $\mathcal{X}$ and $\mathcal{Y}$ be normed spaces and let

$$S : D(S) \subset \mathcal{X} \longrightarrow \mathcal{Y}, \quad T : D(T) \subset \mathcal{X} \longrightarrow \mathcal{Y}$$

be linear operators acting from $\mathcal{X}$ to $\mathcal{Y}$. The *gap from S to T* is defined by

$$\delta(S, T) = \delta(G(S), G(T)), \tag{2.3}$$

and the *gap between S and T* is defined by

$$\hat{\delta}(S, T) = \hat{\delta}(G(S), G(T)). \tag{2.4}$$

More explicitly,

$$\delta(S, T) = \sup_{\substack{u \in D(S) \\ \|u\|_{\mathcal{X}}^2 + \|Su\|_{\mathcal{Y}}^2 = 1}} \inf_{v \in D(T)} \left(\|u - v\|_{\mathcal{X}}^2 + \|Su - Tv\|_{\mathcal{Y}}\right)^{1/2}. \tag{2.5}$$

If S and T are closable operators with closures $\bar{S}$ and $\bar{T}$, then

$$\delta(S, T) = \delta(\bar{S}, \bar{T}). \tag{2.6}$$

Theorem 2.5 *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces and let S, T be densely defined linear operators. Then*

$$\delta(S, T) = \delta(T^*, S^*), \quad \hat{\delta}(S, T) = \hat{\delta}(S^*, T^*), \tag{2.7}$$

where S^*, T^* are the adjoint (*or dual*) *operators of S and T respectively.*

Corollary 2.6 *Let $\mathcal{X}$ be a Hilbert space, and let S, T be essentially selfadjoint* (*in particular, selfadjoint*) *operators in $\mathcal{X}$. Then*

$$\delta(S, T) = \delta(T, S) = \hat{\delta}(S, T). \tag{2.8}$$

Example 2.7 Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces and let $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. It is easily seen that

$$\delta(T, 0) = \frac{\|T\|}{(1 + \|T\|^2)^{1/2}}.$$

In virtue of Theorem 2.5 it follows also that $\delta(0, T) = \delta(T, 0) = \hat{\delta}(T, 0)$. Further results with regards to bounded operator can be found in [6].

The following statement is used to obtain spectral stability estimates. It contains a little improvement compared with the analogous result contained in [9, Chapter IV, p. 204, Theorem 2.17]. For this reason, a slightly modified proof of this result is given here.

Theorem 2.8 *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces,*

$$S : D(S) \subset \mathcal{X} \longrightarrow \mathcal{Y}, \quad T : D(T) \subset \mathcal{X} \longrightarrow \mathcal{Y}$$

be linear operators and let $A \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then

$$\delta(S + A, T + A) \le (2 + \|A\|^2)\delta(S, T), \tag{2.9}$$

Proof. Since $D(A) = \mathcal{X}$, we have $D(S + A) = D(S)$ and $D(T + A) = D(T)$. Let $u \in D(S)$ and

$$\|u\|^2_{\mathcal{X}} + \|(S + A)u\|^2_{\mathcal{Y}} = 1.$$

Then

$$\|Su\|_{\mathcal{Y}} \le \|(S + A)u\|_{\mathcal{Y}} + \|Au\|_{\mathcal{Y}} \le \|(S + A)u\|_{\mathcal{Y}} + \|A\|\|u\|_{\mathcal{X}},$$

hence, by Cauchy-Schwarz inequality,

$$\|Su\|^2_{\mathcal{Y}} \le \big(\|u\|^2_{\mathcal{X}} + \|(S + A)u\|^2_{\mathcal{Y}}\big)\big(1 + \|A\|^2\big) = 1 + \|A\|^2$$

and

$$\big(\|u\|^2_{\mathcal{X}} + \|Su\|^2_{\mathcal{Y}}\big)^{1/2} \le \big(2 + \|A\|^2\big)^{\frac{1}{2}}.$$

Moreover, for any $v \in D(T)$,

$$\begin{aligned} \|(S + A)u - (T + A)v\|_{\mathcal{Y}} &= \|Su - Tv + A(u - v)\|_{\mathcal{Y}} \\ &\le \|Su - Tv\|_{\mathcal{Y}} + \|A\|\|u - v\|_{\mathcal{X}} \end{aligned}$$

hence

$$\|(S + A)u - (T + A)v\|^2_{\mathcal{Y}} \le \big(\|u - v\|^2_{\mathcal{X}} + \|Su - Tv\|^2_{\mathcal{Y}}\big)\big(1 + \|A\|^2\big)$$

and

$$\begin{aligned} &\big(\|u - v\|^2_{\mathcal{X}} + \|(S + A)u - (T + A)v\|^2_{\mathcal{Y}}\big)^{1/2} \\ &\quad \le \big(2 + \|A\|^2\big)^{1/2}\big(\|u - v\|^2_{\mathcal{X}} + \|Su - Tv\|^2_{\mathcal{Y}}\big)^{\frac{1}{2}}. \end{aligned}$$

Therefore

$$\begin{aligned} \delta(S + A, T + A) &= \sup_{\substack{u \in D(S) \\ \|u\|^2_{\mathcal{X}} + \|(S+A)u\|^2_{\mathcal{Y}} = 1}} \inf_{v \in D(T)} \big(\|u - v\|^2_{\mathcal{X}} + \|(S + A)u - (T + A)v\|^2_{\mathcal{Y}}\big)^{1/2} \\ &\le (2 + \|A\|^2)^{1/2} \sup_{\substack{u \in D(S) \\ (\|u\|^2_{\mathcal{X}} + \|Su\|^2_{\mathcal{Y}})^{1/2} \le \\ (2 + \|A\|^2)^{1/2}}} \inf_{v \in D(T)} \big(\|u - v\|^2_{\mathcal{X}} + \|Su - Tv\|^2_{\mathcal{Y}}\big)^{1/2} \\ &= \big(2 + \|A\|^2\big)\delta(S, T). \end{aligned}$$

□

The gap is invariant under inversion, that is, if S, T are invertible operators, then

$$\delta\big(S^{-1}, T^{-1}\big) = \delta(S, T).$$

Theorem 2.9 *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces and let T be an invertible operator with the inverse $T^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$. If S is a closed operator such that*

$$\hat{\delta}(S, T) < \big(1 + \|T^{-1}\|^2\big)^{-1/2},$$

then S is invertible, $S^{-1} \in \mathcal{B}(\mathcal{Y}, \mathcal{X})$ and

$$\big\|S^{-1} - T^{-1}\big\| \le \frac{1 + \big\|T^{-1}\big\|^2}{1 - (1 + \|T^{-1}\|^2)^{1/2}\delta_{\min}(S, T)}\delta_{\min}(S, T),$$

where

$$\delta_{\min}(S, T) = \min\{\delta(S, T),\ \delta(T, S)\}.$$

2.3 Some spectral stability results

If $T : D(T) \subset \mathcal{X} \to \mathcal{X}$ is a closed linear operator, let $\sigma(T)$, $\sigma_p(T)$, $\rho(T)$ denote in order its spectrum, point spectrum (the set of all eigenvalues), and resolvent set respectively. If $\lambda \in \sigma_p(T)$, $N(T-\lambda)$ is the eigenspace of λ.

Let $\mathcal{X}$ be an Hilbert space and let T be a selfadjoint (or even normal) linear operator. Assume that $\sigma(T)$ is separated into two parts $\sigma_1(T)$ and $\sigma_2(T)$ by a positively oriented rectifiable simple closed curve Γ in such a way that it encloses an open set containing $\sigma_1(T)$ in its interior and $\sigma_2(T)$ in its exterior and let

$$P[T] = -\frac{1}{2\pi i}\int_\Gamma (T-\xi)^{-1}d\xi. \tag{2.10}$$

It is well-known that $P[T]$ does not depend on the curve Γ, that is, if Γ' is another positively oriented rectifiable simple curve that encloses an open set containing $\sigma_1(T)$ in its interior and $\sigma_2(T)$ in its exterior, then the second member of (2.10), with Γ' instead of Γ, yields the same operator $P[T]$. It is also known that $P[T]$ is an orthogonal projector. We call the subspace

$$N[T] = P[T]\mathcal{X} \tag{2.11}$$

of $\mathcal{X}$ the *spectral subspace of T corresponding to the part $\sigma_1(T)$ of $\sigma(T)$*. If $\sigma_1(T)$ consists of a finite number of isolated eigenvalues of T, say $\sigma_1[T] = \{\lambda_1, \dots, \lambda_p\}$, then the following relation holds

$$N[T] = \bigoplus_{j=1}^{p} N(T-\lambda_j). \tag{2.12}$$

A sufficient condition for $\sigma_1(T)$ to consist of a finite number of eigenvalues of T is that $N[T]$ has finite dimension. In this case it follows from (2.12) that the sum of the multiplicities of the said eigenvalues equals the dimension of $N[T]$. The proof of these facts is not difficult and can be found in [18, Theorem XII.5] and in [8].

These considerations allow to obtain the following abstract spectral stability result, upon which the stability estimates for the spectral subspaces of nonnegative selfadjoint second-order elliptic operators subject to homogeneous Dirichlet boundary conditions with respect to a perturbation of an open set obtained in the paper rely.

Theorem 2.10 *Let T be a selfadjoint linear operator in $\mathcal{X}$ and let Γ be a positively oriented rectifiable simple closed curve separating the spectrum of T into two parts $\sigma_1(T)$ and $\sigma_2(T)$. Then there exist c, $\delta > 0$ such that, for each selfadjoint linear operator S in $\mathcal{X}$ satisfying $\delta(S, T) < \delta$, this curve also separates the spectrum of S into two parts $\sigma_1(S)$ and $\sigma_2(S)$ and*

$$\hat{\delta}(N[S], N[T]) \le c\delta(S, T). \tag{2.13}$$

Moreover, if $\sigma_1(T)$ consists of a finite number of eigenvalues of T of finite multiplicities, then $\sigma_1(S)$ also consists only of a finite number of eigenvalues of S of finite multiplicities and the sum of the multiplicities of all such eigenvalues of S is equal to the sum of multiplicities of the eigenvalues constituting $\sigma_1(T)$.

Proof. A point $\xi \in \Gamma$ belongs to the resolvent set of S if $S-\xi$ is invertible with the inverse in $\mathcal{B}(\mathcal{X})$, and this, in virtue of Theorems 2.8, 2.9 and Corollary 2.6, is true if $\hat{\delta}(S-\xi, T-\xi) \le \big(2+|\xi|^2\big)\delta(S, T) < \big(1+\|(T-\xi)^{-1}\|^2\big)^{-1/2}$. Therefore, if

$$\delta(S, T) < \delta = \frac{1}{2\max_{\xi\in\Gamma}\,(2+|\xi|^2)\big(1+\|(T-\xi)^{-1}\|^2\big)^{1/2}},$$

then $\Gamma \subset \rho(S)$.

Let $l(\Gamma)$ denote the length of the curve Γ. Using (2.10) and Theorem 2.9, we obtain

$$\begin{aligned}\|P[S]-P[T]\| &= \frac{1}{2\pi}\left\|\int_\Gamma \left\|(S-\xi)^{-1}-(T-\xi)^{-1}\right\|d\xi\right\| \\ &\le \frac{l(\Gamma)}{2\pi}\max_{\xi\in\Gamma}\left\|(S-\xi)^{-1}-(\xi-T)^{-1}\right\| \\ &\le \frac{l(\Gamma)}{2\pi}\max_{\xi\in\Gamma}\frac{1+\left\|(T-\xi)^{-1}\right\|^2}{1-\left(1+\|(T-\xi)^{-1}\|^2\right)^{1/2}\delta_{\min}(S-\xi,\,T-\xi)}\delta_{\min}(S-\xi,\,T-\xi) \\ &\le \frac{l(\Gamma)}{2\pi}\frac{\max_{\xi\in\Gamma}\left(2+|\xi|^2\right)\left(1+\|(T-\xi)^{-1}\|^2\right)}{1-\max_{\xi\in\Gamma}\left(2+|\xi|^2\right)\left(1+\|(T-\xi)^{-1}\|^2\right)^{1/2}\delta(S,\,T)}\delta(S,\,T) \\ &\le c\delta(S,\,T)\end{aligned}$$

with $c = l(\Gamma)/(2\pi\delta)$. Then the result follows from Theorem 2.2.

The second part of the theorem is a consequence of Theorem 2.3 and the considerations preceding this theorem. In order to be able to apply Theorem 2.3 we take also $\delta(S,\,T) < 1/c$. □

Remark 2.11 The previous result can be extended to normal operators by replacing $\delta(S,\,T)$ with $\hat{\delta}(S,\,T)$.

In particular situations it is possible to give a more explicit formula for the constant δ (and hence also for c) in Theorem 2.10. For example, if $\sigma_1(T) = \{\lambda_0\}$ and Γ is a circle centered at λ_0 of radius r that does not enclose any other point of $\sigma(T)$ except λ_0 and that any other point of $\sigma(T)$ different from λ_0 has distance to λ_0 not less than $2r$ (this assumption implies that $\operatorname{dist}(\xi,\,\sigma(T)) = |\xi-\lambda_0| = r$ for all $\xi\in\Gamma$ and therefore $\left\|(T-\xi)^{-1}\right\| = 1/\operatorname{dist}(\xi,\,\sigma(T)) = 1/r$ for each $\xi\in\Gamma$), one can take

$$\delta = \frac{r}{2[2+(|\lambda_0|+r)^2]\,(1+r^2)^{1/2}}.$$

Ultimalety, we apply Theorem 2.10 to unbounded selfadjoint linear operators with compact resolvents bounded from below so we specialize the result to these operators. First we need to introduce some notation in line with (1.6), (1.7). An unbounded selfadjoint linear operator T which is bounded from below and has compact resolvent has an orthonormal basis of eigenfunctions $\{\varphi_k[T]\}_{k\in\mathbb{N}}$ satisfying $T\varphi_k[T] = \lambda_k[T]\varphi_k[T]$ for a sequence of eigenvalues $\{\lambda_k[T]\}_{k\in\mathbb{N}}$ ordered in nondecreasing order (counting multiplicities, $\lambda_k[T]\to\infty$ as $k\to\infty$). Given $k,\,m\ge 1$ such that

$$\lambda_{k-1}[T] < \lambda_k[T] \le \cdots \le \lambda_{k+m-1}[T] < \lambda_{k+m}[T], \tag{2.14}$$

let

$$N_{k,\,m}[T] = \operatorname{span}\{\varphi_k[T],\ldots,\varphi_{k+m-1}[T]\}. \tag{2.15}$$

We have the following

Corollary 2.12 *Let T be an unbounded selfadjoint linear operator with compact ressolvent bounded from below and let $k,\,m\in\mathbb{N}$ be such that* (2.14) *holds. Then there exist $\delta,\,c>0$ such that for all unbounded selfadjoint linear operators with compact resolvent S bounded from below with $\delta(S,\,T)<\delta$ one has*

$$\hat{\delta}(N_{k,\,m}[S],\,N_{k,\,m}[T]) \le c\delta(S,\,T). \tag{2.16}$$

P r o o f. Consider a curve Γ that encloses the only points $\lambda_k[T],\ldots,\lambda_{k+m-1}[T]$ of $\sigma[T]$. In order to complete the proof we need only observe that Γ encloses only the points $\lambda_k[S],\ldots,\lambda_{k+m-1}[S]$ of $\sigma[S]$ for sufficiently small $\delta(S,\,T)$. And this follows from the fact that the $\lambda_i[T]$, $i\in\mathbb{N}$, are continuous functions of T as T varies in the class of selfadjoint linear operators with compact resolvents bounded from below endowed with the gap metric (see the inequalities (2.17) below which hold under the additional assumption that T and S are positive operators).

It is possible to give an alternative proof of this fact without using the so-called Riesz formula (2.10). But first let us abbreviate the notation by calling $\lambda_i[T]$, $\varphi_i[T]$, $\lambda_i[S]$, $\varphi_i[S]$, respectively λ_i, φ_i, λ_i', φ_i', $i\in\mathbb{N}$. Let also

P, Q be the orthogonal projectors of $\mathcal{X}$ onto $N_{k,m}[T]$ and $N_{k,m}[S]$ respectively. For simplicity we suppose that the operators T, S are positive, that is $\lambda_1, \lambda_1{}' > 0$ (recall that in such a case $\|T^{-1}\| = \lambda_1{}^{-1}$). By the Weyl's inequalities and Example 2.7 for all $i \in \mathbb{N}$

$$\left|\lambda_i^{-1} - \lambda_i'^{-1}\right| \le \left\|T^{-1} - S^{-1}\right\| = \frac{\delta\big(T^{-1} - S^{-1}, 0\big)}{\sqrt{1 - \delta\big(T^{-1} - S^{-1}, 0\big)^2}}.$$

On the other hand by Theorem 2.8

$$\begin{aligned}\delta\big(T^{-1} - S^{-1}, 0\big) &\le \big(2 + \|T^{-1}\|^2\big)\delta(T, S)\\ &= \big(2 + \lambda_1{}^{-2}\big)\delta(T, S),\end{aligned}$$

therefore for all $i \in \mathbb{N}$

$$\left|\lambda_i^{-1} - \lambda_i'^{-1}\right| \le \left\|T^{-1} - S^{-1}\right\| \le 2\big(2 + \lambda_1{}^{-2}\big)\delta(T, S) \tag{2.17}$$

provided that $\delta(S, T) < \lambda_1^2/\big(2\lambda_1^2 + 1\big)$. Let us take

$$\delta = \frac{\lambda_1^2}{2\lambda_1^2 + 1}\min\{3/4,\, d_{k,m}/4\}, \tag{2.18}$$

where

$$d_{k,m} = \min\left\{\lambda_{k-1}^{-1} - \lambda_k^{-1},\, \lambda_{k+m-1}^{-1} - \lambda_{k+m}^{-1}\right\};$$

then, by (2.17), $\delta(S, T) < \delta$ implies $\left|\lambda_i^{-1} - \lambda_i'^{-1}\right| < d_{k,m}/2$, $i \in \mathbb{N}$. Hence, for $i = 1, \dots, m$ and $j \ge k + m$, $\left|\lambda_{k+i-1}^{-1} - \lambda_j'^{-1}\right| \ge \lambda_{k+i-1}^{-1} - \lambda_j^{-1} - \left|\lambda_j^{-1} - \lambda_j'^{-1}\right| \ge \lambda_{k+m-1}^{-1} - \lambda_{k+m}^{-1} - \left|\lambda_j^{-1} - \lambda_j'^{-1}\right| > d_{k,m}/2$. Analogously, if $j < k$, $\left|\lambda_{k+i-1}^{-1} - \lambda_j'^{-1}\right| > d_{k,m}/2$. Therefore for $i = 1, \dots, m$ we have

$$\begin{aligned}\left\|T^{-1} - S^{-1}\right\|^2 &\ge \left\|\big(T^{-1} - S^{-1}\big)\varphi_{k+i-1}\right\|^2 = \left\|\lambda_{k+i-1}^{-1}\varphi_{k+i-1} - S^{-1}\varphi_{k+i-1}\right\|^2\\ &= \left\|\lambda_{k+i-1}^{-1}\sum_{j=1}^{\infty}\big(\varphi_{k+i-1}, \varphi_j'\big)\varphi_j' - \sum_{j=1}^{\infty}\lambda_j'^{-1}\big(\varphi_{k+i-1}, \varphi_j'\big)\varphi_j'\right\|^2\\ &= \sum_{j=1}^{\infty}\big(\lambda_{k+i-1}^{-1} - \lambda_j'^{-1}\big)^2\left|\big(\varphi_{k+i-1}, \varphi_j'\big)\right|^2\\ &\ge \sum_{\substack{j=1\\ j \ne k, \dots, k+m-1}}^{\infty}\big(\lambda_{k+i-1}^{-1} - \lambda_j'^{-1}\big)^2\left|\big(\varphi_{k+i-1}, \varphi_j'\big)\right|^2\\ &\ge \frac{d_{k,m}^2}{4}\|(1 - Q)\varphi_{k+i-1}\|^2.\end{aligned}$$

Thus

$$\|(1 - Q)P\| \le \frac{2m}{d_{k,m}}\left\|T^{-1} - S^{-1}\right\|.$$

Analogously, one proves that

$$\|(1 - P)Q\| \le \frac{2m}{d_{k,m}}\left\|T^{-1} - S^{-1}\right\|.$$

Therefore by Theorem 2.2 we have

$$\hat{\delta}(N_{k,m}[S], N_{k,m}[T]) \le \frac{2m}{d_{k,m}}\left\|T^{-1} - S^{-1}\right\|$$

and by (2.17), and (2.18) we obtain

$$\hat{\delta}(N_{k,\,m}[S],\, N_{k,\,m}[T]) \leq \frac{m}{\delta}\delta(S,\, T). \qquad \square$$

In order to apply the previous result to obtain estimates for the eigenvectors we need also the following

Lemma 2.13 *Let $\mathcal{M}$ and $\mathcal{N}$ be finite dimensional subspaces of a Hilbert space $\mathcal{X}$, $\dim \mathcal{M} = \dim \mathcal{N} = m$, and let $u_1, \dots, u_m$ be an orthonormal basis for $\mathcal{M}$. Then there exists an orthonormal basis $v_1, \dots, v_m$ for $\mathcal{N}$ such that*

$$\|u_k - v_k\| \leq 5^k \hat{\delta}(\mathcal{M},\, \mathcal{N}), \quad k = 1, \dots, m. \tag{2.19}$$

P r o o f. Lemma 5.4. of [2] states precisely what we want to prove, with the only distinction that we have (2.19) with $\|P - Q\|$, where P and Q are the orthogonal projectors of $\mathcal{X}$ onto $\mathcal{M}$ and $\mathcal{N}$ respectively, instead of $\hat{\delta}(\mathcal{M},\, \mathcal{N})$. But these quantities are equal in virtue of Theorem 2.2, hence we have the desired result. $\square$

Combining the above results we obtain the following theorem.

Theorem 2.14 *Let T be a selfadjoint linear operator with compact resolvent bounded from below and let $k, m \in \mathbb{N}$ be such that* (2.14) *holds. Then there exist $c, \delta > 0$ such that for each selfadjoint operator with compact resolvent S bounded from below for which $\delta(S,\, T) < \delta$, there exists an orthonormal set of eigenvectors $\varphi'_k[T], \dots, \varphi'_{k+m-1}[T]$ of T corresponding to the eigenvalues $\lambda_k[T], \dots, \lambda_{k+m-1}[T]$ such that*

$$\left\|\varphi_{k+i-1}[S] - \varphi'_{k+i-1}[T]\right\| \leq c\delta(S,\, T) \tag{2.20}$$

for each $i = 1, \dots, m$.

Remark 2.15 The previous result is not symmetric in the operators T, S in the sense that we are obliged to fix first the eigenvectors of S and then we can choose eigenvectors of T (as near to those of S as we wish provided that $\delta(S, T)$ is sufficiently small), but not conversely. Let $\lambda_{k-1}[T] < \lambda_k[T] = \cdots = \lambda_{k+m-1} < \lambda_{k+m}[T]$ for some $k, m \in \mathbb{N}$. The fact is that the subspaces

$$N[T] = N(T - \lambda_k[T]), \quad \text{and} \quad N[S] = \bigoplus_{i=1}^{m} N(S - \lambda_{k+i-1}[S])$$

are indeed near to each other provided that $\hat{\delta}(S, T)$ is sufficiently small (see Theorem 2.10), but we do not have the necessary freedom in choosing the eigenvectors of S since the eigenvalues of S, $\lambda_k[S], \dots, \lambda_{k+m-1}[S]$ are distinct in general (the multiplicity of an eigenvalue of a linear operator T is not stable under small perturbations of T).

2.4 Direct sum of operators

Let $\mathcal{X}$, $\mathcal{X}'$ be normed spaces. Any subspace $\mathcal{M} \subset \mathcal{X}$ can be seen as a subspace of $\mathcal{X} \times \mathcal{X}'$ by identifying it with

$$\{(u, 0) \ : \ u \in \mathcal{M}\};$$

analogously any subspace $\mathcal{M}' \subset \mathcal{X}'$ will be identified with the subspace

$$\{(0, u') \ : \ u' \in \mathcal{M}'\}$$

of $\mathcal{X} \times \mathcal{X}'$. If $u \in \mathcal{X}$ and $u' \in \mathcal{X}'$, we see u and u' as elements of $\mathcal{X} \times \mathcal{X}'$ by identifying them with $(u, 0)$ and $(0, u')$ respectively. Hence

$$(u,\, u') = (u,\, 0) + (0,\, u') \equiv u + u'.$$

Definition 2.16 Let $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}$, $\mathcal{Y}'$ be normed spaces and let

$$S : D(S) \subset \mathcal{X} \longrightarrow \mathcal{Y}, \quad S' : D(S') \subset \mathcal{X}' \longrightarrow \mathcal{Y}'$$

be linear operators. Then the *direct sum operator* $S \dotplus S'$ of S and S' is defined in the following way:

$$D(S \dotplus S') = D(S) \dotplus D(S'),$$

and

$$(S \dotplus S')(u + u') = Su + S'u'$$

for all $u \in D(S)$, $u' \in D(S')$.

If $\mathcal{X}$, $\mathcal{X}'$, $\mathcal{Y}$, $\mathcal{Y}'$ are Hilbert spaces we write $S \oplus S'$ instead of $S \dotplus S'$. In the next proposition some easy-to-prove properties of the direct sum operator are listed.

Proposition 2.17

(i)

$$N(S \dotplus S') = N(S) \dotplus N(S'), \quad R(S \dotplus S') = R(S) \dotplus R(S').$$

(ii)

$$G(S \dotplus S') = G(S) \dotplus G(S').$$

(iii) *$S \dotplus S'$ is invertible if, and only if, S, S' are both invertible and, in this case,*

$$(S \dotplus S')^{-1} = S^{-1} \dotplus S'^{-1}.$$

(iv) *$S \dotplus S'$ is a closable operator if, and only if, S, S' are closable operators and in this case*

$$\overline{S \dotplus S'} = \overline{S} \dotplus \overline{S'}.$$

(v) *$S \dotplus S'$ is a densely defined operator if, and only if, S, S' are densely defined operators and in this case*

$$(S \dotplus S')^* = S^* \dotplus S'^*.$$

(vi) *$S \dotplus S'$ is a bounded* (*respectively, compact*) *operator if, and only if, both S and S' are bounded* (*respectively, compact*) *operators.*

(vii)

$$\sigma(S \dotplus S') = \sigma(S) \cup \sigma(S') \quad \rho(S \dotplus S') = \rho(S) \cap \rho(S').$$

(viii)

$$\sigma_p(S \dotplus S') = \sigma_p(S) \cup \sigma_p(S'),$$

and for any $\lambda \in \mathbb{C}$,

$$N(S \dotplus S' - \lambda I_{\mathcal{X} \dotplus \mathcal{X}'}) = N(S - \lambda I_{\mathcal{X}}) \dotplus N(S' - \lambda I_{\mathcal{X}'}),$$

hence, in particular, the geometrical multiplicity of λ as an eigenvalue of $S \dotplus S'$ is equal to the sum of the geometrical multiplicities of λ as an eigenvalue of S and S' (*this assertion is valid with the understanding that if λ is not an actual eigenvalue of an operator, then λ is looked at as an eigenvalue of geometric multiplicity zero of the said operator*).

Now let $\mathcal{X} = \mathcal{Y}$, $\mathcal{X}' = \mathcal{Y}'$ be Hilbert spaces.

(ix) *$S \oplus S'$ is selfadjoint* (*essentially selfadjoint*) *if, and only if, both S, S' are selfadjoint* (*essentially selfadjoint*) *operators.*

(x) *If S, S' are nonnegative symmetric densely defined linear operators and S^F, ${S'}^F$ are their Friedrich extensions* (*see, for example,* [7, Section 4.4]), *then*

$$(S \oplus S')^F = S^F \oplus {S'}^F,$$

where $(S \oplus S')^F$ is the Friedrich extension of $S \oplus S'$.

3 Spectral stability estimates for the eigenfunctions

As we have already said, we obtain the spectral stability estimates for the variation of the eigenfunctions upon a perturbation of the open set Ω by using the notion of a gap between linear operators. This requires an appropriate extension of the notion of a gap to linear operators determined by different open sets. In the present paper we make use of the definition of a gap that we have proposed in an our earlier work [4]. For convenience we formulate the definition here.

Assume that to each element Ω of a family of bounded open subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, whose boundaries have Lebesgue measure zero a linear operator

$$S_\Omega : D(S_\Omega) \subset L^2(\Omega) \longrightarrow L^2(\Omega) \tag{3.1}$$

is associated, where the linear subspace $D(S_\Omega)$ of $L^2(\Omega)$ is the domain of the operator S_Ω. The reason why we consider only open sets with boundaries of zero Lebesgue measure is that, given two domains $\Omega_1,\ \Omega_2$, we need to identify the spaces $L^2(\Omega_1) \oplus L^2(\Omega_2 \setminus \overline{\Omega_1})$ and $L^2(\Omega_2) \oplus L^2(\Omega_1 \setminus \overline{\Omega_2})$ (which are always identifiable with $L^2(\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1}))$ and $L^2(\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1}))$ respectively) with $L^2(\Omega_1 \cup \Omega_2)$. Indeed, note that this is possible since [1] $\Omega_1 \cup \Omega_2 = (\Omega_1 \cup (\Omega_2 \setminus \overline{\Omega_1})) \cup (\Omega_2 \cap \partial\Omega_1) = (\Omega_2 \cup (\Omega_1 \setminus \overline{\Omega_2})) \cup (\Omega_1 \cap \partial\Omega_2)$ with $|\Omega_2 \cap \partial\Omega_1| = |\Omega_1 \cap \partial\Omega_2| = 0$.

It is useful to make the convention that to the empty set $\emptyset$ the null operator acting from the null space $\{0\}$ with values in $\{0\}$ is associated, so that $S_\Omega \oplus S_\emptyset = S_\Omega$. We assume also that whenever Ω_1, Ω_2 belong to the set of indices of the given family of linear operators (3.1), so does $\Omega_1 \setminus \overline{\Omega_2}$; thus for such Ω_1, Ω_2 we can also consider the operators $S_{\Omega_1 \setminus \overline{\Omega_2}}$ and $S_{\Omega_2 \setminus \overline{\Omega_1}}$.

With these notation we recall the following definition from [4].

Definition 3.1 Let Ω_1, Ω_2 be bounded open sets of $\mathbb{R}^n$ such that $|\partial\Omega_i| = 0$, $i = 1, 2$. The *gap from S_{Ω_1} to S_{Ω_2}* is defined by

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \delta\big(S_{\Omega_1} \oplus S_{\Omega_2 \setminus \overline{\Omega_1}},\ S_{\Omega_2} \oplus S_{\Omega_1 \setminus \overline{\Omega_2}}\big), \tag{3.2}$$

and the *gap between S_{Ω_1} and S_{Ω_2}* is defined by

$$\hat{\delta}(S_{\Omega_1}, S_{\Omega_2}) = \max\{\delta(S_{\Omega_1}, S_{\Omega_2}),\ \delta(S_{\Omega_2}, S_{\Omega_1})\}, \tag{3.3}$$

where the gaps on the right-hand sides of the above equations are gaps from-to and between operators acting in the space $L^2(\Omega_1 \cup \Omega_2)$ according to Definition 2.4.

Remark 3.2 If in the previous definition each of the operators S_{Ω_1}, S_{Ω_2}, $S_{\Omega_2 \setminus \overline{\Omega_1}}$, $S_{\Omega_1 \setminus \overline{\Omega_2}}$ is selfadjoint (or essentially selfadjoint), then the operators $S_{\Omega_1} \oplus S_{\Omega_2 \setminus \overline{\Omega_1}}$, $S_{\Omega_2} \oplus S_{\Omega_1 \setminus \overline{\Omega_2}}$ are selfadjoint (or, respectively, essentially selfadjoint). Thus the gap from S_{Ω_1} to S_{Ω_2} is order-independent, that is

$$\delta(S_{\Omega_1}, S_{\Omega_2}) = \delta(S_{\Omega_2}, S_{\Omega_1}) = \hat{\delta}(S_{\Omega_2}, S_{\Omega_1}),$$

see Proposition 2.17 (ix) and Theorem 2.6.

The application we have in mind when introducing the previous definition is the following. For each bounded open set Ω let $S_\Omega : D(S_\Omega) \subset L^2(\Omega) \to L^2(\Omega)$ be the linear operator introduced in Section 1. In the rest of the paper we use the notation introduced in Section 1. The main result of the present paper is the following

Theorem 3.3 *Fix a bounded open set $\Omega \subset \mathbb{R}^n$, and let $k, m \in \mathbb{N}$ be such that inequalities* (1.7) *hold. Then there exist $c, \delta > 0$ such that*

$$\hat{\delta}(N_{k,m}[\Omega'],\ N_{k,m}[\Omega]) \le c\delta(S_{\Omega'}, S_\Omega) \tag{3.4}$$

for all bounded open sets $\Omega' \subset \mathbb{R}^n$ for which $\delta(S_{\Omega'}, S_\Omega) < \delta$ and $|\Omega \,\Delta\, \Omega'| < \delta$.

P r o o f. The result is a consequence of Corollary 2.12 and of the properties of the direct sum of operators Proposition 2.17. $|\Omega \,\Delta\, \Omega'|$ needs to be sufficiently small in order to ensure that none of $\lambda_{k+i}[\Omega]$, $i = 0, \dots, m-1$,

[1] We denote by $|A|$ the Lebesgue measure of a Lebesgue measurable set $A \subset \mathbb{R}^n$.

is an eigenvalue of $S_{\Omega'\setminus\overline{\Omega}}$ and none of the $\lambda_{k+i}[\Omega']$, $i=0,\dots,m-1$, is an eigenvalue of $S_{\Omega\setminus\overline{\Omega'}}$. This is possible because $\lambda_1[\Omega\setminus\overline{\Omega'}]$, $\lambda_1[\Omega'\setminus\overline{\Omega}]\to\infty$ as $|\Omega\,\Delta\,\Omega'|\to 0$. Thus the spectral subspaces of the operators $S_\Omega\oplus S_{\Omega\setminus\overline{\Omega'}}$, $S_{\Omega'}\oplus S_{\Omega\setminus\overline{\Omega}}$ corresponding to the said eigenvalues coincide with the spectral subspaces of the operators S_Ω, $S_{\Omega'}$ respectively if $|\Omega\,\Delta\,\Omega'|$ is sufficiently small. □

The usefulness of the previous Theorem 3.3 depends on our ability to estimate the gap on the right-hand side of (3.4). We dealt with this problem in our paper [4] and we formulate here the results that we obtained in that work.

If $\Omega\in C^{1,1}$ then the restriction of the operator S_Ω in $D_\Omega=\{u\in C^2(\Omega)\cap C^1(\overline{\Omega})\ :\ u|_{\partial\Omega}=0, Su\in L^2(\Omega))\}$ (using a Green identity one shows that $D_\Omega\subset D(S_\Omega)$) is essentially selfadjoint, that is $S_\Omega=\overline{S_\Omega|_{D_\Omega}}$, e.g., see a comment in Section 3 of [17]. The regularity assumption on Ω is not far from being optimal, for if, e.g., Ω is a non-convex plane polygon, $S_\Omega|_{D_\Omega}$ with $a_{ij}=\delta_{ij}$, $i,j=1,\dots,n$, $b=0$ is not essentially selfadjoint. Indeed, by [3] its closure $\overline{S_\Omega}$ has deficiency indices $(d_-,\,d_+)=(d,\,d)$, where d is the number of non-convex corners.

Hence, if $\Omega_1,\Omega_2,\Omega_1\setminus\overline{\Omega_2},\Omega_1\setminus\overline{\Omega_1}\in C^{1,1}$, then from (2.6) and Proposition 2.17 (iv), (x) it follows that

$$\hat{\delta}(S_\Omega,\,S_{\Omega'})=\delta(S_\Omega,\,S_{\Omega'})=\delta\big(S_\Omega|_{D_\Omega},\,S_{\Omega'}|_{D_{\Omega'}}\big). \tag{3.5}$$

With these observations and notation the following result is a direct consequence of [4, Theorem 4.2].

Theorem 3.4 *Let $n=2,3$ and $\gamma=1/2$ if $n=3$, $0<\gamma<1$ if $n=2$. Let $\Omega\in C^{1,1}$, then there exists a constant $c>0$ such that*

$$\delta(S_\Omega,\,S_{\Omega'})\le c\varepsilon^\gamma \tag{3.6}$$

for all $\varepsilon>0$ and for all open sets $\Omega'\in C^{1,1}$ such that $\Omega_\varepsilon\subset\Omega'\subset\overline{\Omega'}\subset\Omega$.

Using these facts and observations we can deduce the following spectral stability results.

Theorems 3.3 and 3.4 imply

Theorem 3.5 *Let $n=2,3$ and $\gamma=1/2$ if $n=3$, $0<\gamma<1$ if $n=2$. Fix $\Omega\in C^{1,1}$ and let $k,m\in\mathbb{N}$ be such that inequalities (1.7) hold. Then there exist $c,\varepsilon_0>0$ such that*

$$\hat{\delta}(N_{k,m}[\Omega'],\ N_{k,m}[\Omega])\le c\varepsilon^\gamma.$$

if $0<\varepsilon<\varepsilon_0$ and $\Omega'\in C^{1,1}$ is such that $\Omega_\varepsilon\subset\Omega'\subset\overline{\Omega'}\subset\Omega$,.

This theorem together with Lemma 2.13 imply the following

Theorem 3.6 *Let $n=2,3$ and let $\gamma=1/2$ if $n=3$, $0<\gamma<1$ if $n=2$. Fix $\Omega\in C^{1,1}$. Let $\lambda[\Omega]$ be an eigenvalue of multiplicity m ($m\in\mathbb{N}$) of S_Ω, that is, there exists $k\in\mathbb{N}$ such that $\lambda[\Omega]=\lambda_k[\Omega]=\cdots=\lambda_{k+m-1}[\Omega]$. Then there exist $c,\varepsilon_0>0$ such that the following holds: if $0<\varepsilon<\varepsilon_0$ and $\Omega'\in C^{1,1}$ is such that $\Omega_\varepsilon\subset\Omega'\subset\overline{\Omega'}\subset\Omega$ and $\varphi_{k+1}[\Omega'],\dots,\varphi_{k+m}[\Omega']$ is an orthonormal set of eigenfunctions of $S_{\Omega'}$ corresponding to the eigenvalues $\lambda_k[\Omega']\le\cdots\le\lambda_{k+m-1}[\Omega']$, then there exists an orthonormal set of eigenfunctions $\varphi_k[\Omega],\dots,\varphi_{k+m-1}[\Omega]$ of S_Ω corresponding to the eigenvalue $\lambda[\Omega]$ such that*

$$\|\varphi_{k+i}[\Omega]-\varphi_{k+i}[\Omega']\|_{L^2(\Omega')}\le c_0\varepsilon^\gamma$$

for all $i=0,\dots,m-1$.

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