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# Estimates for the deviation of solutions and eigenfunctions of second-order elliptic Dirichlet boundary value problems under domain perturbation

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**Abstract**

Estimates in suitable Lebesgue or Sobolev norms for the deviation of solutions and eigenfunctions of second-order uniformly elliptic Dirichlet boundary value problems subject to domain perturbation in terms of natural distances between the domains are given. The main estimates are formulated via certain natural and easily computable "atlas" distances for domains with Lipschitz continuous boundaries. As a corollary, similar estimates in terms of more "classical" distances such as the Hausdorff distance or the Lebesgue measure of the symmetric difference of domains are derived. Sharper estimates are also proved to hold in smoother classes of domains.


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## 1. Introduction

In this paper we prove stability estimates for solutions and eigenfunctions of second order uniformly elliptic Dirichlet boundary value problems subject to domain perturbation: we give explicit estimates in $L^p$-norm and $W^{1,p}$-norm, where $p$ takes values in a suitable subinterval of $[1, \infty]$, which contains 2 as an interior point, of the difference of solutions and eigenfunc-

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tions on different domains of the Euclidean $n$-dimensional space in terms of suitable distances between the domains such as, e.g., the Hausdorff distance, the Lebesgue measure of the symmetric difference of domains, or even certain atlas distances between domains introduced in the sequel.

In order to describe more precisely our results, let us introduce some notation. Consider a positive symmetric uniformly elliptic linear second-order differential operator in divergence form

$$Su = -\mathrm{div}(A(x)\nabla u) + b(x)u \tag{1.1}$$

in $\mathbb{R}^n$, $n \in \mathbb{N}$, with locally $C^{1,\alpha}$ $(0 < \alpha \leq 1)$ coefficients. That is, we are assuming the matrix $A(x)$ entries and $b(x)$ are locally $C^{1,\alpha}$ functions of $x \in \mathbb{R}^n$ for some $0 < \alpha \leq 1$, such that $A(x)$ is Hermitian and, for a suitable $\lambda > 0$,

$$A(x) \geq \lambda I_n$$

holds (in the sense of the order relation on Hermitian matrices, $I_n$ is the $n$-dimensional unit matrix) for all $x \in \mathbb{R}^n$. In addition, we assume $b(x) \geq 0$ for all $x \in \mathbb{R}^n$.

The estimates regarding solutions that we obtain are of the following type: we find or exhibit examples of

- $\mathcal{F}$, a family of domains (bounded nonempty open sets) in $\mathbb{R}^n$,
- $d(\cdot, \cdot)$ a distance on $\mathcal{F}$,
- $D$, a universal fixed domain that contains all elements of $\mathcal{F}$,
- $\mathcal{G}$ a subfamily of $\mathcal{F}$, to be interpreted as the collection of domains which are being perturbed,
- $\mathcal{G}' = \{\mathcal{G}'_\Omega\}_{\Omega\in\mathcal{G}}$, where, for each $\Omega \in \mathcal{G}$, $\mathcal{G}'_\Omega$ is a subfamily of $\mathcal{F}$, consisting of the so-called *admissible* perturbations of $\Omega$,[1]
- $\mathcal{X}(D)$, $\mathcal{W}(D)$ normed spaces of (possibly, generalized) functions defined on $D$, such that for each $f \in \mathcal{W}(D)$, there exists a unique solution (in some sense[2]) $u_\Omega \in \mathcal{X}(D)$ to problem

$$\begin{cases} Su = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.2}$$

- a parameter $0 < \gamma$ $(\leq 1)$,

such that theorems of the following kind may be formulated.

[1] At a first reading one may take $\mathcal{G} = \mathcal{F}$ and $\mathcal{G}_\Omega = \mathcal{F}$ for all $\Omega \in \mathcal{G} = \mathcal{F}$. However, such an assumption would imply that our stability result (Theorem 1.1) would be symmetric in $\Omega$ and its perturbation $\Omega'$. But this is not always the case: there are results in which (i) we are forced to restrict to a subclass $\mathcal{G}$ of $\mathcal{F}$ of domains whose perturbations we may investigate and (ii) for any $\Omega \in \mathcal{G}$, the class of admissible perturbations of $\Omega$ is not the whole $\mathcal{F}$ but rather a subclass $\mathcal{G}_\Omega$, depending on $\Omega$ itself.

[2] Usually, it is required that $u$ satisfy equation $Su = f$ in $\Omega$ in the sense of distributions, while the boundary values be attained in the sense of traces of Sobolev spaces theory. To be more precise, the problem is uniquely solved by $u_\Omega$ in some normed space $\mathcal{X}(\Omega)$ (depending on $\Omega$), and after extending $u_\Omega$ to all of $D$ (in this paper by setting $u_\Omega = 0$ on $D \setminus \Omega$), then $u_\Omega \in \mathcal{X}(D)$.

**Theorem 1.1.** *There exists $c > 0$ (that depends on $\mathcal{F}$, $\mathcal{G}$, $d(\cdot, \cdot)$, $\mathcal{X}(D)$, $\mathcal{W}(D)$, $\gamma$) such that*

$$\|u_\Omega - u_{\Omega'}\|_{\mathcal{X}(D)} \le c\, d(\Omega, \Omega')^\gamma \|f\|_{\mathcal{W}(D)} \tag{1.3}$$

*for all $\Omega \in \mathcal{G}$, $\Omega' \in \mathcal{G}_\Omega$ and $f \in \mathcal{W}(D)$.*

It seems that there are not many results of this kind in the literature or at least there are no systematic treatments available. One main reason could be that for the shape analysis of many numerical quantities of interest one may usually avoid a preliminary analysis of the dependence of solutions on the domain. However, Proposition 3.3.6 in the book of A. Henrot [1] or the paper of G. Savaré and G. Schimperna [2] provide estimates of the type (1.3) via the Hausdorff distance. Indeed, the results of this paper may seen be as a complementation of those results.

It is known that if $\mathcal{F}$ is the family of all domains in $\mathbb{R}^n$ contained in some fixed domain $D$, then the solution $u_\Omega$ does not depend continuously on $\Omega$ in any reasonable sense. Therefore we must impose geometrical or topological constraints on the family of domains $\mathcal{F}$. Extensive accounts of the necessary and sufficient conditions for the continuous dependence of solutions on the domain can be found in [3,1,4,5]. This theme is beyond the scope of this paper, yet having some knowledge about it helps to form an idea about the kind of results to be expected, the geometrical or topological constraints to be imposed on $\mathcal{F}$ and the kind of distances to be used.

Our working assumption in this paper is that $\mathcal{F}$ consists of domains having boundaries with a uniform Lipschitz continuous character, that is, the boundaries of all the elements of $\mathcal{F}$ are described locally, up to isometric change of coordinates, via the same atlas, as subgraphs of Lipschitz continuous functions with Lipschitz norm not exceeding some positive constant fixed in advance. The proximity of two domains of $\mathcal{F}$ is quantified via certain atlas distances (see Subsect. 2.4 for precise definitions) which can be related fairly easily with more classical distances such as the Lebesgue measure of the symmetric difference or the Hausdorff distance.

Of course, we may very well be interested on domain perturbation stability estimates for Dirichlet problems with inhomogeneous boundary data: for a large class of such problems, we may reduce to the case of a problem with zero boundary values like (1.2), by applying the usual trick of extending the boundary data to all the domain (via a trace theorem), and then changing the unknown by subtracting from it this extension; see [2, Section 3, Corollary 4] for an example with more details.

In order to introduce the second problem we tackle in this article let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$ and consider the following Dirichlet eigenvalue problem

$$\begin{cases} Su = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.4}$$

defined on $\Omega$. The second objective of this paper is to give explicit quantitative stability estimates for the deviation of eigenfunctions of this problem as a result of perturbations of $\Omega$ in terms of suitable distances between the domains that "measure" or "quantify" the size of the said perturbations. The problem (1.4) has a standard weak formulation (which is recalled at the beginning of Subsection 3.4) which leads to a positive selfadjoint operator

$$S_\Omega : D(S_\Omega) \subset L^2(\Omega) \to L^2(\Omega) \tag{1.5}$$

with compact resolvent. In this weak formulation problem (1.4) is the problem of finding the eigenvalues and eigenfunctions of the operator $S_\Omega$.

The issue of the stability of eigenfunctions requires a clarification since eigenfunctions are not uniquely defined and, moreover, the multiplicity of an eigenvalue is not generally stable upon perturbations of the operator $S_\Omega$ (which, in our case, are due to perturbations of the underlying domain $\Omega$). However, we do have the following kind of stability. Consider an eigenvalue of multiplicity two (this paper deals only with operators for which the algebraic and geometric multiplicities of an eigenvalue always coincide, and that is why we speak here only of "the multiplicity" of an eigenvalue) and which therefore has a two-dimensional eigenspace. Usually, when one perturbs "a little" the operator, the said eigenvalue splits (bifurcates) into two "nearby" eigenvalues of the perturbed operator whose eigenspaces are both one-dimensional. Nevertheless, the direct sum of these two eigenspaces is "near" the eigenspace of the original (unperturbed) operator, in the sense that e.g., the angle between these planes is "small". Our objective in this paper is precisely to estimate (the sine of) this angle in terms of suitable distances between the domains.

Let us give a more precise formulation of the problem. Let $\{\lambda_k[\Omega]\}_{k=1}^\infty$ be the sequence of eigenvalues of $S_\Omega$ listed in ascending order and repeated according to multiplicities, and let $\{\varphi_k[\Omega]\}_{k=1}^\infty$ be a sequence of corresponding eigenfunctions chosen in such a way that $\{\varphi_k[\Omega]\}_{k=1}^\infty$ form an orthonormal basis of $L^2(\Omega)$. Let $k, m \in \mathbb{N}$ and suppose that

$$\lambda_{k-1}[\Omega] < \lambda_k[\Omega] \le \cdots \le \lambda_{k+m-1}[\Omega] < \lambda_{k+m}[\Omega]. \tag{1.6}$$

Let

$$N_{k,m}[\Omega] = \operatorname{span}\{\varphi_k[\Omega], \ldots, \varphi_{k+m-1}[\Omega]\}. \tag{1.7}$$

We call these spaces *spectral subspaces* since they are direct sums of eigenspaces (of a finite number of consecutive eigenvalues).

The purpose of the paper is to estimate the change of $N_{k,m}[\Omega]$ in terms of perturbations of $\Omega$. More precisely, we want to find $\mathcal{F}$, $d(\cdot,\cdot)$, $\mathcal{G}$, $\mathcal{G}'$, $D$, $\mathcal{X}(D)$, $\gamma$ as above with[3]

$$N_{k,m}[\Omega] \subset \mathcal{X}(D) \qquad \forall \Omega \in \mathcal{G}$$

such that theorems of the following kind may be formulated.

**Theorem 1.2.** *Given $\Omega \in \mathcal{G}$ and $k, m \in \mathbb{N}$ such that inequalities (1.6) hold, then there exist $c, \delta > 0$ (that depend on $\mathcal{F}$, $d(\cdot,\cdot)$, $\mathcal{G}$, $\mathcal{G}'$, $D$, $\mathcal{X}(D)$, $\gamma$, $\Omega$, $k$, $m$) such that*

$$\hat{\delta}_{\mathcal{X}(D)}(N_{k,m}[\Omega], N_{k,m}[\Omega']) \le c\, d(\Omega, \Omega')^\gamma, \tag{1.8}$$

*for all $\Omega' \in \mathcal{G}'_\Omega$ provided that $d(\Omega, \Omega') \le \delta$.*

[3] This inclusion means that functions in $N_{k,m}[\Omega]$ belong to $\mathcal{X}(D)$ when extended by zero outside $\Omega$ (more generally, we may assume the existence of a bounded linear extension operator $E_{\Omega,k,m} : N_{k,m}[\Omega] \to \mathcal{X}(D)$; with a usual slight abuse of notation, we do not distinguish between $N_{k,m}[\Omega]$ and its image $E_{\Omega,k,m}N_{k,m}[\Omega]$ ).

Here

$$\hat{\delta}_{\mathcal{X}(D)}(N_{k,m}[\Omega],\ N_{k,m}[\Omega'])$$

(defined precisely in §2.1) should be seen as (the sine of) an angle between $N_{k,m}[\Omega]$ and $N_{k,m}[\Omega']$ (considered) as subspaces of $\mathcal{X}(D)$. We stress that the constants $c$, $\delta$ may depend on the domain $\Omega$ and of course on $k$ and $m$, but are independent of $\Omega'$. The exponent $0 < \gamma \le 1$ in (1.8) is independent of both the domains $\Omega, \Omega' \in \mathcal{F}$ and of $k$ and $m$. It is desirable to give sharp values for $\gamma$, in the sense that it is not possible to replace $\gamma$ with some $\gamma' > \gamma$ (unless, e.g., we restrict ourselves to a proper subfamily $\mathcal{G}''$ of $\mathcal{G}'$).

Some results obtained here are listed below (i.e., assigning to $\mathcal{F}$, $d(\cdot,\cdot)$, $D$, $\mathcal{G}$, $\mathcal{G}' = \{\mathcal{G}_\Omega\}_{\Omega\in\mathcal{G}}$, $\mathcal{X}(D)$, $\mathcal{W}(D)$, $\gamma$ the values prescribed below, then Theorems 1.1 and 1.2 hold true).

- $\mathcal{F}$ is a family of bounded Lipschitz domains with uniform Lipschitz character (Lipschitz domains, as it is known, are characterized in terms of a cone condition; one requires that the aperture and the height of the said cone be chosen the same for all domains in $\mathcal{F}$) contained in some fixed bounded domain $D$, $\mathcal{X}(D) = L^2(D)$, $\mathcal{W}(D) = B^{-1/2}_{2,1}(D)$, $d(\cdot,\ \cdot) = d_{\mathcal{H}}(\cdot,\cdot)$ is the Hausdorff distance, $\gamma = 1$ (here $\mathcal{G} = \mathcal{F}$, $\mathcal{G}_\Omega = \mathcal{F}$ for all $\Omega \in \mathcal{G}$).
- $\mathcal{F}, d(\cdot,\cdot), D, (\mathcal{G}, \mathcal{G}_\Omega$ for all $\Omega \in \mathcal{G})$, $\mathcal{W}(D)$ are as above, $X(D) = H^1(D)$, $\gamma = 1/2$. Actually, for solutions, these first two results are due to G. Savaré and G. Schimperna [2].
- $D$ is any fixed bounded domain in $\mathbb{R}^n$, $\mathcal{F}$ the family of all subdomains of $D$, $\mathcal{G}$ the subfamily of $\mathcal{F}$ whose elements have $C^1$ boundary, for all $\Omega \in \mathcal{G}$, $\mathcal{G}_\Omega$ is the subfamily of $\mathcal{F}$ whose elements are subsets of $\Omega$–in other words only inner perturbations of a domain $\Omega$ are being considered in this result–$d(\cdot,\cdot)$ is the Hausdorff distance, $\gamma = 1$, $X(D) = L^\infty(D)$, $\mathcal{W}(D) = L^p(D)$ for any $p > n$. For solutions and $S = -\Delta$ this result is [1, Proposition 3.3.6].
- $\mathcal{F} = C^{0,1}_M(\mathcal{A})$ (by $C^{0,1}_M(\mathcal{A})$ (or $C^1_M(\mathcal{A})$, $C^2_M(\mathcal{A})$, etc.), where $\mathcal{A}$ is an atlas, that is, a finite collection of cuboids in $\mathbb{R}^n$, and $M \ge 0$, one denotes the set of domains whose part of the boundary lying in each of the cuboids of $\mathcal{A}$ is–up to an isometric change of coordinates–the graph of a Lipschitz (or $C^1$, $C^2$, etc.) function with Lipschitz (or $C^1$, $C^2$, etc.) norm $\le M$), $d(\cdot,\ \cdot) = d_{\mathcal{A},r}(\cdot,\ \cdot)$ a so-called *atlas distance* (which is defined by taking the supremum (or the sum) of the $L^r$-norms of the differences of the functions describing the boundaries of two domains in each of the cuboids of $\mathcal{A}$, see §2.4 for precise definitions), $D$ a fixed bounded domain that contains all elements of $\mathcal{F}$, $X(D) = L^q(D)$, $\mathcal{W}(D) = W^{s-1,p}(D)$, with $r = (1 + s - 1/p)(1/q - 1/p)^{-1}$, $\gamma = 1 + s - 1/p$, where $0 \le s \le 1$, $1 \le \bar{q}(\mathcal{F}, s) < q \le p < \bar{p}(\mathcal{F}, s) \le \infty$ (here $\bar{q}(\mathcal{F}, s) < 2$, $\bar{p}(\mathcal{F}, s) > 2$ are parameters that depend only on $\mathcal{F}$ and $s$, we have also $\bar{p}(\mathcal{F}, 0) > 3$), $1 \le p < 1/s$ (if $p = 2$ one may reach also $s = 1/2$), $\mathcal{W}(D) = W^{s-1,1}(D)$ (or, for $p = 2$, $s = 1/2$, $\mathcal{W}(D) = B^{2,1}_{-1/2}(D)$).
- If in the previous bullet we restrict to $\mathcal{F} = C^1_M(\mathcal{A})$, we can take $\bar{p}(\mathcal{F}, s) = 1$, $\bar{p}(\mathcal{F}, s) = \infty$.
- $\mathcal{F} = C^2_M(\mathcal{A})$ for some given atlas $\mathcal{A}$ and constant $M \ge 0$, $d(\cdot,\ \cdot) = d_{\mathcal{A},q}(\cdot,\ \cdot)$, $D$ is a bounded domain that contains all elements of $\mathcal{F}$, $\mathcal{X}(D) = L^q(D)$, $\mathcal{W}(D) = L^p(D)$ for any $1 \le q \le \infty$, $p > n$, and $\gamma = 1$.

For the results in the last three bullets see Theorem 3.2 for solutions and §3.4 for eigenfunctions. In all these results $\gamma$ is sharp.

It is worth emphasizing that $d_{\mathcal{A},r}(\cdot,\ \cdot)$ are weaker than the Hausdorff distance, and they allow one to keep better track of the local variations of domains. For example, in the case of

$\Omega, \Omega' \in C^{0,1}_M(\mathcal{A})$ for some atlas $\mathcal{A}$ and constant $M \geq 0$, taking $q = 2$, $p = 3$, $s = 0$ in the forth bullet above, one obtains the estimate

$$\|u_\Omega - u_{\Omega'}\|_{L^2(D)} \leq c\, d_{\mathcal{H}}(\Omega, \Omega')^{1/2} \cdot |\Omega \,\Delta\, \Omega'|^{1/6} \|f\|_{L^3(D)},$$

where $d_{\mathcal{H}}(\cdot, \cdot)$ is the Hausdorff distance–for any measurable set $A$, $|A|$ denotes its Lebesgue measure–and $f \in L^3(D)$. Suppose that $\Omega \,\Delta\, \Omega'$ is contained in some small ball of radius $\varepsilon$. Form the estimate above one deduces $\|u_\Omega - u_{\Omega'}\|_{L^2(D)} \leq c\, \varepsilon^{1/2+n/6} \|f\|_{L^3(D)}$ which is a finer estimate, at least for dimensions $n \geq 4$, than the estimate $\|u_\Omega - u_{\Omega'}\|_{L^2(D)} \leq c\, \varepsilon \|f\|_{L^2(D)} \leq c\, \varepsilon \|f\|_{L^3(D)}$, deriving from estimates expressed only in terms of the Hausdorff distance (as in the first bullet above or as in [2]).

Actually, if $\Omega, \Omega' \in C^2_M(\mathcal{A})$, taking $q = 2$, and some $p > n$ in the last bullet we obtain the finer estimate

$$\|u_\Omega - u_{\Omega'}\|_{L^2(D)} \leq c\, d_{\mathcal{A},2}(\Omega, \Omega') \|f\|_{L^p(D)} \leq c\, d_{\mathcal{H}}(\Omega, \Omega')^{1/2} \cdot |\Omega \,\Delta\, \Omega'|^{1/2} \|f\|_{L^p(D)}.$$

In this paper we prove estimates for the deviation of solutions and eigenfunctions in $W^{1,q}$-norm as well via an interpolation technique.

The paper is organized as follows. In §2 we fix notation and present preliminary results and definitions that are used throughout the paper. In §2.1 the notion of a gap between subspaces and its properties are recalled; in §2.2 and in §2.3 theorems allowing to derive estimates for spaces of eigenfunctions from estimates for the resolvent operators (that is, solutions) are presented in a Hilbert and a Banach space context, respectively. (Although standard these results are presented here in such a fashion that they be readily applicable to operators arising from domain perturbation problems as explained in §3.4. The proof of Theorem 2.5 may be new.) In §2.4 the notions of atlas and atlas distances are introduced, and properties relating these atlas distances with each other, the Hausdorff distance and the Lebesgue measure of the symmetric difference of domains are presented. §3 contains the main results of the paper: in §3.1 we prove stability estimates in $L^p$-norm for the deviation of solutions under domain perturbation; in §3.2, using the previous estimates and an interpolation technique we derive stability estimates in $W^{1,p}$-norm; in §3.3 the sharpness of the exponent $\gamma$ is proved; and finally, in §3.4, making use of the "abstract" stability results for eigenfunctions outlined in the previous section, we provide stability estimates for eigenfunctions under domain perturbation.

## 2. Notation, background information, spectral stability estimates

### 2.1. Gap between subspaces

We begin this section by recalling the definition of a gap between subspaces of a normed space and collecting some of its properties that are used throughout the paper. Heuristically, the gap between two subspaces may be seen as the "sine of the smallest angle" between the said subspaces (in particular, it assumes values between 0 and 1).

If $\mathcal{M}, \mathcal{N}$ are linear subspaces of a Banach space $\mathcal{X}$, the *gap from* $\mathcal{M}$ to $\mathcal{N}$ is defined by the following formula:

$$\delta(\mathcal{M}, \mathcal{N}) = \delta_{\mathcal{X}}(\mathcal{M}, \mathcal{N}) = \sup_{\substack{u \in \mathcal{M} \\ \|u\|=1}} \operatorname{dist}(u, \mathcal{N}), \tag{2.1}$$

where $\operatorname{dist}(u, \mathcal{N}) = \inf_{v \in \mathcal{N}} \|u - v\|$ is the distance of the vector $u$ to the subspace $\mathcal{N}$. By convention, if $\mathcal{M} = \{0\}$, one defines $\delta(\{0\}, \mathcal{N}) = 0$. One also defines the *gap between* $\mathcal{M}$ and $\mathcal{N}$ by

$$\hat{\delta}(\mathcal{M}, \mathcal{N}) = \hat{\delta}_{\mathcal{X}}(\mathcal{M}, \mathcal{N}) = \max\{\delta(\mathcal{M}, \mathcal{N}), \delta(\mathcal{N}, \mathcal{M})\}. \tag{2.2}$$

We keep the subindex $\mathcal{X}$ in $\delta_{\mathcal{X}}(\mathcal{M}, \mathcal{N})$ or in $\hat{\delta}_{\mathcal{X}}(\mathcal{M}, \mathcal{N})$ only when we wish to emphasize the norm of the space $\mathcal{X}$ in which the gap is being calculated, otherwise we drop it. The gap provides a natural way in which to formulate perturbation estimates about spectral subspaces and eigenfunctions. We need the following facts.

**Proposition 2.1.** *If $\mathcal{M}$ and $\mathcal{N}$ are linear subspaces of a Hilbert space $\mathcal{H}$, then*

$$\begin{aligned} \delta(\mathcal{M}, \mathcal{N}) &= \|(1 - Q)P\| \\ \hat{\delta}(\mathcal{M}, \mathcal{N}) &= \|P - Q\|, \end{aligned} \tag{2.3}$$

*where $P$, $Q$ are the orthogonal projectors of $\mathcal{H}$ onto the closures of $\mathcal{M}$ and $\mathcal{N}$, respectively.*

**Proposition 2.2.** *Let $\mathcal{M}$ and $\mathcal{N}$ be linear subspaces of a Banach space $\mathcal{X}$. If $\dim \mathcal{M} = \dim \mathcal{N} < \infty$, then*

$$\delta(\mathcal{N}, \mathcal{M}) \le \frac{\delta(\mathcal{M}, \mathcal{N})}{1 - \delta(\mathcal{M}, \mathcal{N})}.$$

A simple proof of Proposition 2.1 can be found e.g., in [6] while Proposition 2.2 is proved in [7].

It is also worth pointing out, although we do not use this fact here, that, if $\mathcal{M}$ and $\mathcal{N}$ are subspaces of a Hilbert space $\mathcal{H}$ such that $\hat{\delta}(\mathcal{M}, \mathcal{N}) < 1$, then $\dim \mathcal{M} = \dim \mathcal{N}$ and $\delta(\mathcal{M}, \mathcal{N}) = \delta(\mathcal{N}, \mathcal{M})$. This follows from Proposition 2.1 and [8, Theorem I.6.34, p. 56].

The following lemma, proved in [9] (see also [10]) allows one to derive, in a Hilbert space context, some kind of stability estimates about the eigenfunctions once we have stability estimates about the gap between spectral subspaces.

**Lemma 2.3.** *Let $\mathcal{M}$ and $\mathcal{N}$ be finite dimensional subspaces of a Hilbert space $\mathcal{H}$, $\dim \mathcal{M} = \dim \mathcal{N} = m$, and let $u_1, \ldots, u_m$ be an orthonormal basis for $\mathcal{M}$. Then there exists an orthonormal basis $v_1, \ldots, v_m$ for $\mathcal{N}$ such that*

$$\|u_k - v_k\| \le 5^k \hat{\delta}(\mathcal{M}, \mathcal{N}) \quad \textit{for all } k = 1, \ldots, m. \tag{2.4}$$

*2.2. Spectral stability estimates in a Hilbert space context*

By the notation $T : D(T) \subset \mathcal{X} \to \mathcal{X}$ we denote a linear operator acting in a normed space $\mathcal{X}$ with domain $D(T)$. Let us denote by $N[T]$ and $R[T]$ its kernel and range, respectively.

In this subsection let $\mathcal{X} = \mathcal{H}$ be a Hilbert space and let $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ be a nonnegative selfadjoint unbounded linear operator with compact resolvent when restricted to $R[T]$. Note that $R[T]$ is a closed subspace of $\mathcal{H}$ with orthogonal complement $N[T]$, and that both $N[T]$, $R[T]$

are invariant subspaces of $T$ (in the sense that $T(N[T]) \subset N[T]$, $T\big(D(T) \cap R[T]\big) \subset R[T]$). We are assuming that

$$\hat{T} = T|_{D(T)\cap R[T]} : D(T) \cap R[T] \subset R[T] \to R[T], \qquad u \to Tu$$

is not only bijective with a bounded inverse $\hat{T}^{-1} : R[T] \to R[T]$, but, that in addition, this inverse, which in the sequel we denote simply by

$$T^{-1} : R[T] \to R[T]$$

(with a slight abuse of notation) is a compact operator in $R[T]$. For us it will be convenient to see $T^{-1}$ as an operator acting in all of $\mathcal{H}$, that is,

$$T^{-1} : \mathcal{H} \to \mathcal{H},$$

by setting it zero on $N[T]$ and extending it by linearity on all of $\mathcal{H} = N[T] \oplus R[T]$.

Thus, if $T$ is as stated above,[4] the spectrum of $T$ is a discrete and unbounded set of nonnegative real numbers and its nonzero elements are eigenvalues of finite multiplicity. It is convenient to represent these nonzero eigenvalues of $T$ as a nondecreasing sequence $\{\lambda_k[T]\}_{k=1}^{\infty}$ of positive numbers diverging to infinity, where each eigenvalue is repeated as many times as its multiplicity. Let $\{\varphi_k[T]\}_{k=1}^{\infty}$ be an orthonormal basis of the orthogonal complement of $N[T]$ in $\mathcal{H}$, which is $R[T]$, consisting of eigenvectors of $T$, where each $\varphi_k[T]$ is an eigenvector for the eigenvalue $\lambda_k[T]$. Given $k, m \in \mathbb{N}$ such that

$$\lambda_{k-1}[T] < \lambda_k[T] \le \lambda_{k+1}[T] \le \cdots \le \lambda_{k+m-1}[T] < \lambda_{k+m}[T], \tag{2.5}$$

we define

$$N_{k,m}[T] = \operatorname{span}\{\varphi_k[T], \ldots, \varphi_{k+m-1}[T]\}. \tag{2.6}$$

Actually the definition of $N_{k,m}[T]$ makes sense for any $m, k \in \mathbb{N}$ and we use implicitly this fact in the sequel. But stability estimates are proved only for those subspaces $N_{k,m}[T]$ with $k, m \in \mathbb{N}$ satisfying condition (2.5); we call such subspaces *spectral subspaces* of $T$ as they are direct sums of eigenspaces of $T$ corresponding to a finite number of consecutive eigenvalues of $T$.

Lat us recall a fact that we use, in particular, in Proposition 2.7 below. If $S$ is a nonnegative selfadjoint operator with compact resolvent when restricted to its range, then the norm of its "inverse" (as defined above) is given by

$$\|S^{-1}\| = \lambda_1[S]^{-1}.$$

[4] The reason for dealing with such operators rather than with merely self-adjoint operators with compact resolvent is due to the fact that operators arising from boundary value problems act on different normed spaces that depend on the domain, so that by some procedure, in our case a simple "extension by zero" (see §3.4 for the details), they have to be seen as operators acting on only one fixed normed space, in order to apply the stability theory of this section. But the said procedure makes the kernel of an operator infinite-dimensional and so it cannot have a compact resolvent. Nevertheless, the restriction of such an operator to its range has indeed compact resolvent.

The following lemma constitutes the core of the “abstract” spectral stability estimates, upon which our estimates are based.

**Lemma 2.4.** *Let $T$ be an unbounded nonnegative selfadjoint linear operator with compact resolvent when restricted to its range and let $k, m \in \mathbb{N}$ be such that (2.5) and (2.6) hold. Then there exist $c, \delta > 0$ such that*

$$\delta(N_{k,m}[T],\ N_{k,m}[S]) \le c\,\|(S^{-1} - T^{-1})|_{N_{k,m}[T]}\|$$

*for any unbounded nonnegative selfadjoint linear operator $S$ with compact resolvent when restricted to its range such that*

$$\max\{|\lambda_{k-1}[S]^{-1} - \lambda_{k-1}[T]^{-1}|,\ |\lambda_{k+m}[S]^{-1} - \lambda_{k+m}[T]^{-1}|\} \le \delta. \tag{2.7}$$

**Proof.** Let us shorten the notation: we set $\lambda_i = \lambda_i[T]$, $\varphi_i = \varphi_i[T]$, $\lambda'_i = \lambda_i[S]$, $\varphi'_i = \varphi_i[S]$ for all $i \in \mathbb{N}$. Let us take

$$\delta = \frac{1}{2}\min\{\lambda_{k-1}^{-1} - \lambda_k^{-1},\ \lambda_{k+m-1}^{-1} - \lambda_{k+m}^{-1}\}. \tag{2.8}$$

Hence, by (2.7) for $i = 1, \dots, m$ and $j \ge k+m$, $|\lambda_{k+i-1}^{-1} - \lambda_j'^{-1}| \ge \lambda_{k+i-1}^{-1} - \lambda_{k+m}'^{-1} \ge \lambda_{k+i-1}^{-1} - \lambda_{k+m}^{-1} - |\lambda_{k+m}^{-1} - \lambda_{k+m}'^{-1}| \ge 2\delta - \delta = \delta$. Analogously, if $j < k$, $|\lambda_{k+i-1}^{-1} - \lambda_j'^{-1}| \ge \delta$. Therefore for $i = 1, \dots, m$, since

$$\varphi_{k+i-1} = \sum_{j=1}^{\infty} < \varphi_{k+i-1},\ \varphi'_j > \varphi'_j + v$$

for some $v \in N[S]$ (thus $Sv = 0$ and $< v\,, \varphi'_i >= 0$ for all $i \in \mathbb{N}$) we have

$$\begin{aligned}
\|(T^{-1} - S^{-1})|_{N_{k,m}[T]}\|^2 &\ge \|(T^{-1} - S^{-1})\varphi_{k+i-1}\|^2 = \|\lambda_{k+i-1}^{-1}\varphi_{k+i-1} - S^{-1}\varphi_{k+i-1}\|^2 \\
&= \Bigg\| \lambda_{k+i-1}^{-1} \sum_{j=1}^{\infty} < \varphi_{k+i-1},\ \varphi'_j > \varphi'_j + \lambda_{k+i-1}^{-1} v \\
&\quad - \sum_{j=1}^{\infty} \lambda_j'^{-1} < \varphi_{k+i-1},\ \varphi'_j > \varphi'_j \Bigg\|^2 \\
&= \sum_{j=1}^{\infty} (\lambda_{k+i-1}^{-1} - \lambda_j'^{-1})^2 | < \varphi_{k+i-1},\ \varphi'_j > |^2 + \lambda_{k+i-1}^{-2}\|v\|^2 \\
&\ge \sum_{\substack{j=1 \\ j \ne k,\dots,k+m-1}}^{\infty} (\lambda_{k+i-1}^{-1} - \lambda_j'^{-1})^2 | < \varphi_{k+i-1},\ \varphi'_j > |^2 + \lambda_{k+i-1}^{-2}\|v\|^2 \\
&\ge \min\{\delta^2,\ \lambda_{k+i-1}^{-2}\}\|(1 - Q)\varphi_{k+i-1}\|^2 \\
&= \min\{\delta^2,\ \lambda_{k+i-1}^{-2}\}\|(1 - Q)P\varphi_{k+i-1}\|^2,
\end{aligned}$$

where $P$ and $Q$ denote the orthogonal projectors of $\mathcal{H}$ onto $N_{k,m}[T]$ and $N_{k,m}[S]$, respectively. Thus, by (2.3)

$$\begin{aligned}\delta(N_{k,m}[T], N_{k,m}[S]) &= \|(1-Q)P\| \\ &\leq \frac{m}{\min\{\delta, \lambda_{k+m-1}^{-1}\}}\|(T^{-1}-S^{-1})|_{N_{k,m}[T]}\|. \qquad \square\end{aligned}$$

The previous lemma together with Proposition 2.2 implies the following

**Theorem 2.5.** *Let $T$ be an unbounded nonnegative selfadjoint linear operator with compact resolvent when restricted to its range and let $k, m \in \mathbb{N}$ be such that (2.5) and (2.6) hold. Then there exist $c, \delta > 0$ such that*

$$\hat{\delta}(N_{k,m}[T], N_{k,m}[S]) \leq c\,\|(S^{-1}-T^{-1})|_{N_{k,m}[T]}\| \tag{2.9}$$

*for any unbounded nonnegative selfadjoint linear operator $S$ with compact resolvent when restricted to its range such that*

$$\max\left\{|\lambda_{k-1}[S]^{-1}-\lambda_{k-1}[T]^{-1}|,\ |\lambda_{k+m}[S]^{-1}-\lambda_{k+m}[T]^{-1}|,\ \|(S^{-1}-T^{-1})|_{N_{k,m}[T]}\|\right\} \leq \delta. \tag{2.10}$$

Now a few comments are in order about these spectral stability estimates. First of all, it must be said that results in the spirit of Theorem 2.5 are not new in the literature: compare with [11, Theorem 7.1]. Nevertheless, Theorem 2.5 contains some slight improvements with respect to [11, Theorem 7.1]: first the "size" of the perturbation, that is, the analog of the left-hand side in (2.10), which needs to be sufficiently small for estimates (2.9) to hold, in the letter theorem is simply $\|T^{-1}-S^{-1}\|$ which is, in general, a quantity much larger than the left-hand side of (2.10); second, the proof of Theorem 2.5 seems to be new in the context of Hilbert spaces theory. However, the advantage of [11, Theorem 7.1] is that it is proved via the so-called Riesz formula (2.14) below and holds in a general Banach space. We use this observation in order to extend our estimates to a Banach space context in the forthcoming subsection.

Combining Theorem 2.5 with Lemma 2.3 we obtain some kind of stability estimates for the eigenvectors described in the following

**Theorem 2.6.** *Let $T$ be a nonnegative selfadjoint unbounded linear operator with compact resolvent when restricted to its range and let $k$, $m \in \mathbb{N}$ be such that inequalities (2.5) hold. Then there exist $c$, $\delta > 0$ such that for any nonnegative selfadjoint unbounded linear operator $S$ with compact resolvent when restricted to its range for which (2.7) holds, there exists an orthonormal set of eigenvectors $\varphi_k[T], \ldots, \varphi_{k+m-1}[T]$ of $T$ corresponding to the eigenvalues $\lambda_k[T], \ldots, \lambda_{k+m-1}[T]$ such that*

$$\|\varphi_{k+i-1}[S]-\varphi_{k+i-1}[T]\| \leq c\,\|(S^{-1}-T^{-1})|_{N_{k,m}[T]}\| \tag{2.11}$$

*for each $i = 1, \ldots, m$.*

The spectral stability estimates given above are rather "good" as we show with the next proposition. This means that if we have estimates of the quantity $\|(S^{-1}-T^{-1})|_{N_{k,\,m}[T]}\|$ that are sharp in some sense, and know that the deviation of the eigenvalues tends to zero more rapidly than $\|(S^{-1}-T^{-1})|_{N_{k,\,m}[T]}\|$, which is often the case in applications, then we may derive estimates for $\hat{\delta}(N_{k,\,m}[T],\ N_{k,\,m}[S])$ that are sharp in that same sense.

**Proposition 2.7.** *Let $T$ be an unbounded nonnegative selfadjoint linear operator with compact resolvent when restricted to its range, and let $k, m \in \mathbb{N}$ be such that (2.5) and (2.6) hold. Actually, assume that $\lambda_k[T]$ is an eigenvalue of multiplicity $m$ (that is, $\lambda_k[T] = \cdots = \lambda_{k+m-1}[T]$, thus $N_{k,m}[T]$ is the eigenspace of $\lambda_k[T]$). Let also $\varepsilon > 0$. Then there exists $c > 0$ such that*

$$\|(S^{-1}-T^{-1})|_{N_{k,\,m}[T]}\| \le c \left( \hat{\delta}(N_{k,\,m}[T],\ N_{k,\,m}[S]) + \max_{1\le i\le m} |\lambda_{k+i-1}[S] - \lambda_{k+i-1}[T]| \right) \tag{2.12}$$

*for any unbounded nonnegative selfadjoint linear operator $S$ with compact resolvent when restricted to its range and such that $\lambda_1[S] \ge \varepsilon$.*

**Proof.** We use the same notation as in the proof of Lemma 2.4. Using Lemma 2.3 we may choose $\varphi_{k+i-1} = \varphi_{k+i-1}[T]$, $i = 1, \ldots, m$, in such a way that they satisfy in addition

$$\|\varphi'_{k+i-1} - \varphi_{k+i-1}\| \le 5^i \hat{\delta}(N_{k,\,m}[T],\ N_{k,\,m}[S])\,. \tag{2.13}$$

Let $u \in N_{k,\,m}[T]$ with $\|u\| = 1$, that is, $u = \sum_{l=1}^{m} a_i \varphi_{k+i-1}$ for some $a_i \in \mathbb{C}$, $i = 1, \ldots, m$ with $\sum_{i=1}^{m} |a_i|^2 = 1$. Since

$$\begin{aligned}
(S^{-1}-T^{-1})u &= \sum_{i=1}^{m} a_i S^{-1}\varphi_{k+i-1} - \sum_{i=1}^{m} a_i \lambda_{k+i-1}^{-1}\varphi_{k+i-1} \\
&= \sum_{i=1}^{m} a_i S^{-1}(\varphi_{k+i-1} - \varphi'_{k+i-1}) + \sum_{i=1}^{m} a_i {\lambda'_{k+i-1}}^{-1}(\varphi'_{k+i-1} - \varphi_{k+i-1}) \\
&\quad + \sum_{i=1}^{m} a_i ({\lambda'_{k+i-1}}^{-1} - \lambda_{k+i-1}^{-1})\varphi_{k+i-1},
\end{aligned}$$

we have

$$\|(S^{-1}-T^{-1})u\| \le \sqrt{m}\big(\|S^{-1}\| + {\lambda'_k}^{-1}\big) \max_{i=1,\ldots,m} \|\varphi_i - \varphi'_i\| + \max_{i=1,\ldots,m} |\lambda_{k+i-1}^{-1} - {\lambda'}_{k+i-1}^{-1}|$$

Since ${\lambda'_k}^{-1} \le \|S^{-1}\| \le 1/\varepsilon$, taking also into account (2.13), we obtain the claimed estimate. $\square$

### 2.3. Extension of spectral stability estimates to a Banach space context

As in the previous subsections let $\mathcal{H}$ denote a Hilbert space and let $\mathcal{X}$ be a Banach space. Assume, in addition, that $\mathcal{H}$ and $\mathcal{X}$ are both continuously embedded in some other topological space $\mathcal{E}$. We denote by $\mathcal{F}(\mathcal{H}, \mathcal{X})$ the collection of all nonnegative selfadjoint unbounded linear operators $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ with compact resolvent when restricted to $R[T]$ that satisfy the following conditions: there exists another linear operator $\bar{T} : D(\bar{T}) \subset \mathcal{X} \to \mathcal{X}$ acting in the Banach space $\mathcal{X}$, which $(i)$ is *consistent* with $T$, that is, $\bar{T}u = Tu$ for all $u \in D(T) \cap D(\bar{T})$, $(ii)$ has discrete spectrum, and $(iii)$ has the same eigenvalues and generalized eigenvectors with $T$, that is, if $\varphi$ is a *generalized eigenvector* for $\bar{T}$, which means that $(\bar{T} - \lambda)^p \varphi = 0$ for some $p \in \mathbb{N}$ and some $\lambda \in \mathbb{C}$, then, $\varphi$ is also an eigenvector of $T$ associated to the eigenvalue $\lambda$.

We use the same notation regarding $T$ as in the previous subsection, that is, we denote by $\{\lambda_k[T]\}_{k=1}^{\infty}$ its sequence of positive eigenvalues, listed in ascending order, taking into account multiplicities, and by $\{\varphi_k[T]\}_{k=1}^{\infty}$ an orthonormal basis of the orthogonal complement of $N[T]$ in $\mathcal{H}$, where, for all $k \in \mathbb{N}$, $\varphi_k[T]$ is an eigenvector of $T$ for the eigenvalue $\lambda_k[T]$. Of course, $\{\varphi_k[T]\}_{k=1} \subset D(T) \cap D(\bar{T})$ and $\{\lambda_k[T]\}_{k=1}^{\infty}$ are (with the possible exception of zero) the only eigenvalues of $\bar{T}$.

For any pair of positive integers $k$, $m$ for which (2.5) hold, let $N_{k,m}[T]$ be given by (2.6). Let $\Gamma$ be a rectifiable simple closed curve in the complex plane $\mathbb{C}$ that encloses only $\lambda_k[T], \dots, \lambda_{k+m-1}[T]$ in its interior and the rest of the spectrum of $T$ (or $\bar{T}$) lies in the exterior of the region determined by $\Gamma$. Under these assumptions the following identity holds:

$$N_{k,m}[T] = R[P[\bar{T}]], \quad \text{where} \quad P[\bar{T}] = -\frac{1}{2\pi i}\int_{\Gamma} (\bar{T} - \xi)^{-1} d\xi. \tag{2.14}$$

For the proof of this fact see e.g., [12, Theorem XII.5] and [8] (in particular, Subsections 4 and 5 of Chapter III, Section 6).

Using (2.14) we can prove, in a similar fashion as [11, Theorem 7.1], an analogous of Theorem 2.5 for operators in $\mathcal{F}(\mathcal{H}, \mathcal{X})$, where the gap and the operator norm in the left hand side of (2.9) are calculated in terms of the norm of the Banach space $\mathcal{X}$. More precisely, the result reads as follows.

**Theorem 2.8.** *Let $T \in \mathcal{F}(\mathcal{H}, \mathcal{X})$ and let $k, m \in \mathbb{N}$ be such that (2.5) and (2.6) hold. Then there exist $c, \delta > 0$ such that*

$$\hat{\delta}_{\mathcal{X}}(N_{k,m}[T], N_{k,m}[S]) \le c \,\|(S^{-1} - T^{-1})|_{N_{k,m}[T]}\|_{\mathcal{X}} \tag{2.15}$$

*for any $S \in \mathcal{F}(\mathcal{H}, \mathcal{X})$ such that*

$$\max\left\{|\lambda_{k-1}[S]^{-1} - \lambda_{k-1}[T]^{-1}|,\ |\lambda_{k+m}[S]^{-1} - \lambda_{k+m}[T]^{-1}|,\ \|(S^{-1} - T^{-1})|_{N_{k,m}[T]}\|_{\mathcal{X}}\right\} \le \delta.$$

### 2.4. Different types of domains and atlas distances

For any set $V$ in $\mathbb{R}^n$ and $\delta > 0$ we denote by $V_\delta$ the set $\{x \in V : d(x, \mathbb{R}^n \setminus V) > \delta\}$, and by $V^\delta$ the set $\{x \in \mathbb{R}^n : d(x, V) < \delta\}$.

Given $d > 0$, $\sigma \in \mathbb{N}$, and a family of bounded open cuboids $\{V_j\}_{j=1}^{\sigma}$ and a family $\{r_j\}_{j=1}^{\sigma}$ of rotations in $\mathbb{R}^n$, one says that $\mathcal{A} = \left(d, \sigma, \{V_j\}_{j=1}^{\sigma}, \{r_j\}_{j=1}^{\sigma}\right)$ is an *atlas* in $\mathbb{R}^n$ with parameters $d, \sigma, \{V_j\}_{j=1}^{\sigma}, \{r_j\}_{j=1}^{\sigma}$, briefly an *atlas in* $\mathbb{R}^n$.

Let $\mathcal{N} = \mathcal{N}(\mathbb{R}^{n-1}, \mathbb{R})$ be a family (usually, but not always, a linear space) of real-valued functions on $\mathbb{R}^{n-1}$. One denotes by $\mathcal{N}(\mathcal{A})$ the family of all open sets $\Omega$ in $\mathbb{R}^n$ satisfying the following properties:

(i) $\partial\Omega \subset \bigcup_{j=1}^{\sigma} (V_j)_d$;
(ii) $(V_j)_d \cap \partial\Omega \neq \emptyset$ for $j = 1, \dots \sigma$;
(iii) for $j = 1, \dots, \sigma$

$$r_j(V_j) = \{x \in \mathbb{R}^n : \ a_{ij} < x_i < b_{ij}, \ i = 1, \dots, n\},$$

and

$$r_j(\Omega \cap V_j) = \{x \in \mathbb{R}^n : \ a_{nj} < x_n < g_j(\bar{x}), \ \bar{x} \in W_j\},$$

where $\bar{x} = (x_1, \dots, x_{n-1})$, $W_j = \{\bar{x} \in \mathbb{R}^{n-1} : \ a_{ij} < x_i < b_{ij}, i = 1, \dots, n-1\}$ and $g_j \in \mathcal{N}$; moreover for $j = 1, \dots, \sigma$

$$a_{nj} + d \le g_j(\bar{x}) \le b_{nj} - d \qquad \text{for all } \bar{x} \in \overline{W}_j .$$

If $\Omega \in \mathcal{N}(\mathcal{A})$ one describes the above facts by simply saying that *the boundary of* $\Omega$ *is described by the atlas* $\mathcal{A}$ *and the family of functions* $\{g_j\}_{j=1}^{\sigma}$.

One says indistinguishably that $\Omega$ is a $\mathcal{N}$*-domain*, or that $\Omega$ has an $\mathcal{N}$*-boundary*, or that $\Omega$ is a *domain of class* $\mathcal{N}$ if $\Omega \in \mathcal{N}(\mathcal{A})$ for some atlas $\mathcal{A}$.

Thus, since $C = C(\mathbb{R}^{n-1}, \mathbb{R})$ denotes, as usual, the space of continuous functions on $\mathbb{R}^{n-1}$, $C(\mathcal{A})$ denotes the class of open sets with a *continuous boundary* described by the atlas $\mathcal{A}$. If one denotes by $C_M^{0,1} = C_M^{0,1}(\mathbb{R}^{n-1}, \mathbb{R})$, $M > 0$, the set of Lipschitz continuous functions on $\mathbb{R}^{n-1}$ with Lipschitz constant $\le M$, then $C_M^{0,1}(\mathcal{A})$ denotes the class of open sets with boundaries described as above by the atlas $\mathcal{A}$ and functions $g_j \in C_M^{0,1}(\mathbb{R}^{n-1})$, $j = 1, \dots, \sigma$. The reader can guess the mining of $C_M^1(\mathcal{A})$, $C_M^2(\mathcal{A})$, etc. One says that an open set $\Omega$ in $\mathbb{R}^n$ is an open set with a *continuous boundary* if $\Omega$ is of class $C(\mathcal{A})$ for some atlas $\mathcal{A}$. Analogously, one says that $\Omega$ is an open set with a *Lipschitz boundary*, if $\Omega \in C_M^{0,1}(\mathcal{A})$ for soma atlas $\mathcal{A}$ and some $M > 0$.

Let $\mathcal{A} = \left(d, \sigma, \{V_j\}_{j=1}^{\sigma}, \{r_j\}_{j=1}^{\sigma}\right)$ be an atlas in $\mathbb{R}^N$ and $1 \le r \le \infty$. For all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ we define the *atlas distance* $d_{\mathcal{A},r}(\cdot, \cdot)$ by

$$d_{\mathcal{A},r}(\Omega_1, \Omega_2) = \left( \sum_{j=1}^{\sigma} \left\| g_{1j} - g_{2j} \right\|_{L^r(W_j)}^r \right)^{1/r},$$

where $g_{1j}, g_{2j}$ are the functions describing the boundaries of $\Omega_1, \Omega_2$, respectively. In the case $r = \infty$ this formula should be modified in the usual way (that is, intended as a supremum).

Observe that the function $d_{\mathcal{A},r}(\cdot,\cdot)$ is indeed a distance in $C(\mathcal{A})$ (the distance $d_{\mathcal{A},\infty}(\cdot,\cdot)$ has been introduced and used for obtaining estimates for the deviation of eigenvalues in [13]).

It is easy to verify that if $\Omega_1, \Omega_2 \in C(\mathcal{A})$ for some atlas $\mathcal{A}$, then $\Omega_1 \cap \Omega_2 \in C(\mathcal{A})$ and for all $1 \le r \le \infty$ one has

$$d_{\mathcal{A},r}(\Omega_1 \cap \Omega_2, \Omega_2) \le d_{\mathcal{A},r}(\Omega_1, \Omega_2). \tag{2.16}$$

It is easy to prove the following elementary comparison inequalities between the Lebesgue measure of the symmetric difference of domains and the atlas distances, and among the atlas distances themselves. Below $|\Omega|$ stands for the Lebesgue $n$-measure of a set $\Omega \subset \mathbb{R}^n$.

**Lemma 2.9.** *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^n$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1 |\Omega_1 \Delta \Omega_2| \le d_{\mathcal{A},1}(\Omega_1, \Omega_2) \le c_2 |\Omega_1 \Delta \Omega_2|. \tag{2.17}$$

*Let, in addition, $1 \le r_1 \le r_2 \le r_3 \le \infty$ be such that $1/r_2 = (1-\theta)/r_1 + \theta/r_3$ for some $0 \le \theta \le 1$. Then, there exist $c_1, c_2 > 0$ such that*

$$c_1 d_{\mathcal{A},r_1}(\Omega_1, \Omega_2) \le d_{\mathcal{A},r_2}(\Omega_1, \Omega_2) \le c_2 d_{\mathcal{A},r_1}(\Omega_1, \Omega_2)^{1-\theta} d_{\mathcal{A},r_3}(\Omega_1, \Omega_2)^{\theta} \tag{2.18}$$

*for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$.*

For the definition of the Hausdorff distance $d_{\mathcal{H}}(\Omega_1, \Omega_2)$ between two domains we refer the reader, e.g., to [1], pages 28–29. The following fact can be proved easily.

**Lemma 2.10.** *Let $\mathcal{A}$ be an atlas and let $M > 0$. Then there exist $c, \delta > 0$ (that depend only on $\mathcal{A}$ and $M$) such that for all $\Omega_1, \Omega_2 \in C_M^{0,1}(\mathcal{A})$*

$$d_{\mathcal{A},\infty}(\Omega_1, \Omega_2) \le c\, d_{\mathcal{H}}(\Omega_1, \Omega_2) \tag{2.19}$$

*provided that $d_{\mathcal{H}}(\Omega_1, \Omega_2) \le \delta$.*

Let $\Omega \in C(\mathcal{A})$, $1 \le p \le \infty$. It turns out to be useful to introduce an $L^p$-function space on $\partial\Omega$ that generally depends on the atlas $\mathcal{A}$. Let $u$ be a function defined on $\partial\Omega$, one says that $u \in L^p_{\mathcal{A}}(\partial\Omega)$ if, considered a partition of unity $\{\psi_j\}_{j=1}^s$ of $\partial\Omega$ subordinate to the cover $\{V_j\}_{j=1}^s$, then $(\psi_j u) \circ r_j^{-1}(\cdot, g_j(\cdot)) \in L^p(W_j)$ for all $j = 1, \dots, s$ and we set

$$\|u\|_{L^p_{\mathcal{A}}(\partial\Omega)} = \left( \sum_{j=1}^{s} \|(\psi_j u) \circ r_j^{-1}(\cdot, g_j(\cdot))\|^p_{L^p(W_j)} \right)^{1/p},$$

with the usual modification (i.e., the sum intended as a supremun) when $p = \infty$. One easily verifies that the definition of $L^p_{\mathcal{A}}(\partial\Omega)$ does not depend on the particular partition of unity chosen above (the norms generated by any pair of partitions of unity are equivalent). If $\Omega$ is a Lipschitz domain then of course $L^p_{\mathcal{A}}(\partial\Omega) = L^p(\partial\Omega)$, the Lebesgue space on $\partial\Omega$ defined by means of the standard surface measure on $\partial\Omega$, with equivalent norms.

## 3. Stability estimates for solutions and eigenfunctions

In the first two subsections we present the results of our analysis about the stability of solutions to problem (1.2) under domain perturbation.

Our estimates hold usually in the context of families of domains $\mathcal{F}$ with a uniform Lipschitz character. We first give estimates for the $L^q$-norm of the difference of solutions, where the summability exponent $q$ varies in a suitable subinterval of $[1, \infty]$ containing 2 as an interior point and that depends on $\mathcal{F}$, in terms of suitable distances between domains, and, subsequently, we give also estimates in $W^{1,q}$-norm relying on interpolation techniques and/or suitable boundary decay estimates. In the last subsection, using these same estimates together with Theorem 2.5 or Theorem 2.8 we present similar estimates for the gap between spectral subspaces.

The derivation of the stability estimates for solutions is based, in particular, on deep regularity results of Jerison, Kenig [14], Mitrea, Taylor [15–17] for problem (1.2) in the context of domains $\Omega$ with a Lipschitz continuous boundary in the scale of Sobolev–Besov spaces.

Let us introduce briefly the notation that we use to denote Besov and fractional order Sobolev spaces. Let $(\cdot,\cdot)_{\theta,q}$ be the real interpolation functor [18,19]. For $0 \le s \le 1$, $1 \le p, q \le \infty$, $\Omega \subset \mathbb{R}^n$ open set, we define

$$B_s^{p,q}(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{s,q}, \quad B_{-s}^{p,q}(\Omega) = (L^p(\Omega), W^{-1,p}(\Omega))_{s,q},$$
$$B_{1+s}^{p,q}(\Omega) = (W^{1,p}(\Omega), W^{2,p}(\Omega))_{s,q} = \{u \in W^{1,p}(\Omega) : \nabla u \in B_s^{p,q}(\Omega)\}.$$

Moreover, for $0 < s < 1$, we set

$$W^{s,p}(\Omega) = B_s^{p,p}(\Omega), \quad W^{-s,p}(\Omega) = B_{-s}^{p,p}(\Omega), \quad W^{1+s,p}(\Omega) = B_{1+s}^{p,p}(\Omega).$$

As usual, if $p = 2$, we write $H^s(\Omega)$ instead of $W^{s,2}(\Omega)$. Extensive treatments of the fractional-order Sobolev and Besov spaces theory can be found, e.g., in [20,18,21,19,22]. For the definition and properties of Sobolev and Besov spaces in boundaries of Lipschitz domains see also [14].

### *3.1. Stability estimates for solutions*

For the rest of the paper let $S$ be a fixed second-order uniformly elliptic divergent-form linear differential operator as in (1.1) which satisfies the assumptions mentioned in the introduction. Constants and certain parameters in the estimates below depend usually also on the coefficients of $S$, but we do not point this out explicitly since $S$ is fixed once for all.

As it is clear from the nature of our problem (see the introduction) the first step consists in the identification of well-posedness pairs of function spaces for problem (1.2). For that purpose the have the following well-posedness result for (1.2) due to the deep work of Jerison, Kenig [14], Mitrea, Taylor [15–17].

**Lemma 3.1.** *Let $\mathcal{A}$ be an atlas, $M > 0$ and consider the family of domains $\mathcal{F} = C_M^{0,1}(\mathcal{A})$. Let also $0 \le s \le 1$. Then there exists a maximal interval $]\tilde{q}(\mathcal{F}, s), \tilde{p}(\mathcal{F}, s)[ \subset [1, \infty]$ with $1 \le \tilde{q} = \tilde{q}(\mathcal{F}, s) = 2 < \tilde{p} = \tilde{p}(\mathcal{F}, s) \le \infty$ such that for all $\Omega \in \mathcal{F}$, $\tilde{q} < p < \tilde{p}$, $p < 1/s$ and for all $f \in W^{s-1,p}(\Omega)$ the problem (1.2) has a unique solution $u_\Omega \in W^{1+s,p}(\Omega)$; in addition, this solution satisfies*

$$\|u_\Omega\|_{W^{1+s,\,p}(\Omega)} \le c\,\|f\|_{W^{s-1,\,p}(\Omega)} \tag{3.1}$$

*for some* $c = c(\mathcal{F},\ p,\ s) > 0$.

*The result holds also for* $p = 2$ *and* $s = 1/2$ *provided that* $W^{s-1,\,p}(\Omega)$ *is replaced by* $B^{2,1}_{-1/2}(\Omega)$.

*If* $s = 0$, *then* $\tilde{q}(\mathcal{F},\ 0) > 3$.

*The result holds with* $\tilde{q}(\mathcal{F},\ s) = 1$ *and* $\tilde{p}(\mathcal{F},\ s) = \infty$ *if either the family of domains is* $\mathcal{F} = C^1_M(\mathcal{A})$, *or if* $M$ *is sufficiently small.*

*If* $\mathcal{F} = C^{1,\,1}_M(\mathcal{A})$ *the result holds for any* $1 < p < \infty$ *and* $0 \le s \le 1$.

The following theorem is probably the most important result of the paper, and gives stability estimates for the deviation of solutions in $L^q$-norm, for "$q$"s in a suitable subinterval of $[1, \infty]$ containing 2 as an interior point, as a result of domain perturbation, in terms of suitable distances between the domains, that quantify the said perturbation.

**Theorem 3.2.** *Let* $\mathcal{A}$ *be an atlas,* $M > 0$, *and let* $\mathcal{F} = C^{0,\,1}_M(\mathcal{A})$. *Let* $D$ *be a bounded domain that contains all elements of* $\mathcal{F}$. *Let also* $0 \le s \le 1$. *Then there exists a maximal interval* $]\bar{q},\ \bar{p}[\ =\ ]\bar{q}(\mathcal{F},\ s),\ \bar{p}(\mathcal{F},\ s)[\ \subset [1,\ \infty]$ *containing* 2 *as an interior point and such that for all* $\Omega \in \mathcal{F}$, $\bar{q} < q \le p < \bar{p}$, $p < 1/s$, *and for all* $f \in W^{s-1,\,p}(D)$ *the problem* (1.2) *has a unique solution* $u_\Omega \in W^{1,\,p}(\Omega)$*; moreover, there exists a* $c = c(\mathcal{F},\ p,\ q,\ s) > 0$ *such that*

$$\|u_\Omega - u_{\Omega'}\|_{L^q(D)} \le c\, d_{\mathcal{A},\,r}(\Omega,\ \Omega')^\gamma\, \|f\|_{W^{s-1,\,p}(D)} \tag{3.2}$$

*for all* $\Omega, \Omega' \in \mathcal{F}$, *where*

$$r = \left(1 + s - \frac{1}{p}\right)\left(\frac{1}{q} - \frac{1}{p}\right)^{-1}, \quad \gamma = 1 + s - \frac{1}{p} \tag{3.3}$$

*(*$u_\Omega, u_{\Omega'}$ *are extended trivially to* $D$ *by setting them zero outside their domains of definition).*

*The result holds also for* $p = 2$ *and* $s = 1/2$ *provided that we replace* $W^{s-1,\,p}(D)$ *above with* $B^{2,1}_{-1/2}(\Omega)$.

*If* $s = 0$, *then* $\bar{p}(\mathcal{F}, 0) > 3$.

*The result holds with* $\tilde{q}(\mathcal{F},\ s) = 1$ *and* $\bar{p}(\mathcal{F},\ s) = \infty$ *if either the family of domains is* $\mathcal{F} = C^1_M(\mathcal{A})$, *or if* $M$ *is sufficiently small.*

*If* $\mathcal{F} = C^2_M(\mathcal{A})$, *then the result holds also for* $s = 0$, $p = \infty$ *and for any* $1 \le q \le \infty$ *(with* $\gamma = 1$ *and* $r = q$*) provided that* $W^{s-1,\,p}(D)$ *is replaced by* $L^p(D)$ *for any* $p > n$.

The proof of this theorem relies also on the following lemmas. We begin with a kind of maximum principle for $S$-harmonic functions.

**Lemma 3.3.** *Let* $\mathcal{F} = C^{0,\,1}_M(\mathcal{A})$, *where* $\mathcal{A}$ *is an atlas and* $M > 0$. *Then, there exist* $1 \le q_1 = q_1(\mathcal{F}) < 2$ *and a constant* $c = c(\mathcal{F},\ q) > 0$ *such that for all* $\Omega \in \mathcal{F}$, *for all* $q_1 < q \le \infty$ *and for all* $u \in W^{1,\,q}(\Omega)$, $u$ $S$-harmonic, *that is,* $Su = 0$,

$$\|u\|_{L^q(\Omega)} \le c\,\|u|_{\partial\Omega}\|_{L^q(\partial\Omega)}. \tag{3.4}$$

*For $q=\infty$ we can take $c=1$ above (maximum principle) and actually no regularity assumption on $\Omega$ is needed at all.*

*If either $\mathcal{F}=C^1_M(\mathcal{A})$ or $M$ is sufficiently small, then we can take $q_1=q_1(\mathcal{F})=1$.*

The restriction $u|_{\partial\Omega}$ above should be understood in the sense of traces for $q\le n$. Otherwise, it is just a usual restriction since $u$ is continuous. For the precise meaning of "$Su=0$" and for the proof of this result we refer to the papers of Mitrea and Taylor (which, in particular, extend potential theory to variable coefficients second-order elliptic operators such as $S$ on Lipschitz domains), and, in particular, to [15, Proposition 9.1] and to the subsequent paper [17] in which the authors reduce the regularity assumptions on the coefficients of $S$.

Next, we need the following estimate for the difference of norms of boundary values of a function on different boundaries.

**Lemma 3.4.** *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^n$ and $0\le s\le 1$.*

(i) *If $1\le p<1/s$ and $1\le q\le p$, then there exists $c>0$, depending only on $\mathcal{A}$, $p$, $s$ and $q$, such that for all $\Omega_1,\ \Omega_2\in C(\mathcal{A})$ and for all $u\in W^{1+s,\,p}(\Omega_1\cup\Omega_2)$*

$$\Big|\|u|_{\partial\Omega_1}\|_{L^q_{\mathcal{A}}(\partial\Omega_1)}-\|u|_{\partial\Omega_2}\|_{L^q_{\mathcal{A}}(\partial\Omega_2)}\Big|\le c\,d_{\mathcal{A},\,r}(\Omega_1,\Omega_2)^{1+s-\frac{1}{p}}\|u\|_{W^{1+s,\,p}(\Omega_1\cup\Omega_2)}\,, \tag{3.5}$$

*where*

$$r=\Big(1+s-\frac{1}{p}\Big)\mu\,,\quad \mu=\Big(\frac{1}{q}-\frac{1}{p}\Big)^{-1}.$$

(ii) *If $p=1/s$ and $1\le q\le p$, then for each $\varepsilon\in(0,1)$ there exists $c_\varepsilon>0$, depending only on $\varepsilon$, $\mathcal{A}$, $p$ and $q$, such that for all $\Omega_1,\ \Omega_2\in C(\mathcal{A})$ and for all $u\in W^{1+s,\,p}(\Omega_1\cup\Omega_2)$*

$$\Big|\|u|_{\partial\Omega_1}\|_{L^q_{\mathcal{A}}(\partial\Omega_1)}-\|u|_{\partial\Omega_2}\|_{L^q_{\mathcal{A}}(\partial\Omega_2)}\Big|\le c_\varepsilon\,d_{\mathcal{A},\,(1-\varepsilon)\mu}(\Omega_1,\Omega_2)^{1-\varepsilon}\|u\|_{W^{1+s,\,p}(\Omega_1\cup\Omega_2)}\,. \tag{3.6}$$

(iii) *The estimate above holds also for $\varepsilon=0$ if $s=0,\ p=\infty$.*

(iv) *If $0\le s<\frac{n}{p}$, $1\le q\le\frac{np}{n-sp}$ and $M>0$, then there exists $c>0$, depending only on $\mathcal{A}$, $M$, $p$, $s$ and $q$, such that for all $\Omega_1,\ \Omega_2\in C^{0,1}_M(\mathcal{A})$ and for all $u\in W^{1+s,\,p}(\Omega_1\cup\Omega_2)$*

$$\Big|\|u|_{\partial\Omega_1}\|_{L^q_{\mathcal{A}}(\partial\Omega_1)}-\|u|_{\partial\Omega_2}\|_{L^q_{\mathcal{A}}(\partial\Omega_2)}\Big|\le c\,d_{\mathcal{A},\,\varrho}(\Omega_1,\Omega_2)^{1-1/p+s/n}\|u\|_{W^{1+s,\,p}(\Omega_1\cup\Omega_2)}\,, \tag{3.7}$$

*where*

$$\varrho=\Big(1-\frac{1}{p}+\frac{s}{n}\Big)\nu\,,\quad \nu=\Big(\frac{1}{q}-\frac{1}{p}+\frac{s}{n}\Big)^{-1}.$$

**Proof.** 1. First assume that $d > 0$, $-\infty < a_i < b_i < \infty$, $i = 1, \dots, n-1$,

$$W = \{\bar{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1} : \ a_i < x_i < b_i, \ i = 1, \dots, n-1\},$$
$$\Omega_k = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : \ a_n < x_n < g_k(\bar{x}), \ \bar{x} \in W\}, \quad k = 1, 2, \tag{3.8}$$

where $g_k \in C(\overline{W})$, $k = 1, 2$, are such that

$$a_n + d \le g_k(\bar{x}) \le b_n - d, \quad \bar{x} \in \overline{W}.$$

Moreover, let

$$h_1(\bar{x}) = \min\{g_1(\bar{x}), g_2(\bar{x})\}, \quad h_2(\bar{x}) = \max\{g_1(\bar{x}), g_2(\bar{x})\}, \quad \bar{x} \in \overline{W}.$$

Then

$$\Omega_1 \cup \Omega_2 = \{x \in \mathbb{R}^n : \ a_n < x_n < h_2(\bar{x}), \ \bar{x} \in W\}$$

and

$$d_{\mathcal{A}, r}(\Omega_1, \Omega_2) = \|h_2 - h_1\|_{L^r(W)}.$$

Let $u \in W^{1+s, p}(\Omega_1 \cup \Omega_2)$. Then $u$ is equivalent to a function, which we denote by the same letter, such that for almost all $\bar{x} \in W$ the function $u(\bar{x}, \cdot)$ is absolutely continuous on the interval $[a_n, h_2(\bar{x})]$. By the Newton–Leibnitz formula

$$u(\bar{x}, g_1(\bar{x})) = u(\bar{x}, g_2(\bar{x})) + \int_{g_2(\bar{x})}^{g_1(\bar{x})} \frac{\partial u}{\partial x_n}(\bar{x}, x_n) dx_n \tag{3.9}$$

for all such $\bar{x} \in W$.

By Minkowski's inequality, for any $1 \le q < \infty$, we get

$$\|u(\cdot, g_1(\cdot))\|_{L^q(W)} \le \|u(\cdot, g_2(\cdot))\|_{L^q(W)} + J,$$

where

$$J = \left\| \int_{h_1(\bar{x})}^{h_2(\bar{x})} \left| \frac{\partial u}{\partial x_n}(\bar{x}, x_n) \right| dx_n \right\|_{L^q(W)}.$$

2. If $s = 0$ and $1 \le q \le p$, then to estimate $J$ it suffices to apply twice Hölder's inequality:

$$\begin{aligned} J &\le \left\| (h_2(\bar{x}) - h_1(\bar{x}))^{\frac{1}{p'}} \left\| \frac{\partial u}{\partial x_n}(\bar{x}, \cdot) \right\|_{L^p(h_1(\bar{x}), h_2(\bar{x}))} \right\|_{L^q(W)} \\ &\le \|(h_2(\bar{x}) - h_1(\bar{x}))^{\frac{1}{p'}}\|_{L^\mu(W)} \left\| \left\| \frac{\partial u}{\partial x_n}(\bar{x}, \cdot) \right\|_{L^p(a_n, h_2(\bar{x}))} \right\|_{L^p(W)} \end{aligned} \tag{3.10}$$

since $\frac{1}{\mu}+\frac{1}{p}=\frac{1}{q}$. Hence

$$J \le \|h_2(\bar{x})-h_1(\bar{x})\|^{\frac{1}{p'}}_{L^{\frac{\mu}{p'}}(W)} \left\|\frac{\partial u}{\partial x_n}\right\|_{L^p(\Omega_1\cup\Omega_2)} \le d_{\mathcal{A},\,(1-1/p)\,\mu}(\Omega_1,\Omega_2)^{1-\frac{1}{p}}\|u\|_{W^{1,p}(\Omega_1\cup\Omega_2)}.$$

3. If $1\le p<1/s$ and $1\le q\le p$, then we use the one-dimensional continuous embedding $W^{s,p}(a_n,h_2(\bar{x}))\hookrightarrow L^{q_*}(a_n,h_2(\bar{x}))$, where $q_*=p/(1-sp)$ and the fact that the norm of the embedding operator corresponding to this embedding is bounded above by a quantity $c>0$ depending only on $p,s$ and $d$, see, e.g., [20]. Similarly to the previous step we have

$$\begin{aligned}
J &\le \left\|(h_2(\bar{x})-h_1(\bar{x}))^{\frac{1}{q_*'}}\left\|\frac{\partial u}{\partial x_n}(\bar{x},\cdot)\right\|_{L^{q_*}(h_1(\bar{x}),h_2(\bar{x}))}\right\|_{L^q(W)} && (3.11)\\
&\le c\left\|(h_2(\bar{x})-h_1(\bar{x}))^{\frac{1}{q_*'}}\left\|\frac{\partial u}{\partial x_n}(\bar{x},\cdot)\right\|_{W^{s,p}(h_1(\bar{x}),h_2(\bar{x}))}\right\|_{L^q(W)}\\
&\le c\left\|(h_2(\bar{x})-h_1(\bar{x}))^{\frac{1}{q_*'}}\right\|_{L^{\mu}(W)}\left\|\left\|\frac{\partial u}{\partial x_n}(\bar{x},\cdot)\right\|_{W^{s,p}(a_n,h_2(\bar{x}))}\right\|_{L^p(W)}\\
&= c\left\|h_2(\bar{x})-h_1(\bar{x})\right\|^{\frac{1}{q_*'}}_{L^{\frac{\mu}{q_*'}}(W)}\left\|\frac{\partial u}{\partial x_n}\right\|_{W^{s,p}(\Omega_1\cup\Omega_2)}\\
&\le c\,d_{\mathcal{A},\,\mu/q_*'}(\Omega_1,\Omega_2)^{1/q_*'}\|u\|_{W^{1+s,p}(\Omega_1\cup\Omega_2)} && (3.12)\\
&= c\,d_{\mathcal{A},\,(1-1/p+s)\,\mu}(\Omega_1,\Omega_2)^{1-1/p+s}\|u\|_{W^{1+s,p}(\Omega_1\cup\Omega_2)}.
\end{aligned}$$

If $p=1/s$, we may apply the one-dimensional continuous embedding $W^{s,p}(a_n,h_2(\bar{x}))\hookrightarrow L^{q_*}(a_n,h_2(\bar{x}))$ inequality as above for any $1\le q_*<\infty$, in particular, for $q_*=1/\varepsilon$ in order to derive, by (3.12),

$$J \le c\,d_{\mathcal{A},\,(1-\varepsilon)\,\mu}(\Omega_1,\Omega_2)^{1-\varepsilon}\|u\|_{W^{1+1/p,p}(\Omega_1\cup\Omega_2)}. \tag{3.13}$$

4. There is an alternative way of estimating $J$. Let $1\le p<1/s$. Starting with inequality (3.10), we may use the inequality

$$\|v(\bar{x},\cdot)\|_{L^p(h_1(\bar{x}),h_2(\bar{x}))} \le c\,(h_2(\bar{x})-h_1(\bar{x}))^s\|v\|_{W^{s,p}(h_1(\bar{x}),h_2(\bar{x}))}, \tag{3.14}$$

where $c>0$ depends only on $p,s$ and $d$; see [23, Theorem 5.9, p. 251]. Then similarly to Step 2 we get

$$\begin{aligned}
J &\le c\,\|(h_2(\bar{x})-h_1(\bar{x}))^{\frac{1}{p'}+s}\|_{L^{\mu}(W)}\left\|\left\|\frac{\partial u}{\partial x_n}(\bar{x},\cdot)\right\|_{W^{s,p}(a_n,h_2(\bar{x}))}\right\|_{L^p(W)}\\
&\le c\,d_{\mathcal{A},\,(1-1/p+s)\,\mu}(\Omega_1,\Omega_2)^{1-\frac{1}{p}+s}\|u\|_{W^{1+s,p}(\Omega_1\cup\Omega_2)}. && (3.15)
\end{aligned}$$

5. Let $p=1/s$, then $p<1/(s-\varepsilon)$. Applying inequality (3.15) with $s-\varepsilon$ instead of $s$ we find (3.13).

6. Proof of $(iv)$. As usual, by Hölder's inequality we obtain (3.11), where now we take $q_* = np/(n-sp)$. Since $1/q = 1/\nu + 1/q_*$, applying once more time Hölder's inequality, we obtain

$$J \le \| \, |h_1 - h_2|^{\frac{1}{q'_*}} \|_{L^\nu(W)} \left\| \frac{\partial u}{\partial x_n} \right\|_{L^{q_*}(\Omega_1 \cup \Omega_2)}.$$

Since $sp < n$, we use the $n$-dimensional continuous embedding $W^{s,p}(\Omega_1 \cup \Omega_2) \hookrightarrow L^{q_*}(\Omega_1 \cup \Omega_2)$ whose corresponding operator norm is bounded above by a quantity $c > 0$ that depends only on $n, p, s, \mathcal{A}, M$, in order to obtain

$$J \le c \, d_{\mathcal{A},\varrho}(\Omega_1, \Omega_2)^{1-1/p+\frac{s}{n}} \|u\|_{W^{1+s,p}(\Omega_1 \cup \Omega_2)}.$$

7. We conclude by using a smooth partition of unity $\{\psi_j\}_{j=1}^\sigma$ subordinate to the covering $\{V_j\}_{j=1}^\sigma$ of $\partial\Omega' \cup \partial\Omega$ such that $|D^\alpha \psi_j(x)| \le c$ for all $x \in \mathbb{R}^n$ and for all multiindices $\alpha$ with $|\alpha| \le 2$, $j = 1, \dots, \sigma$, where $c_2 > 0$ depends only on the atlas $\mathcal{A}$. □

Finally, we need also the following Poincaré type inequality.

**Lemma 3.5.** *Let $\mathcal{A} = \left(d, \sigma, \{V_j\}_{j=1}^\sigma, \{r_j\}_{j=1}^\sigma\right)$ be an atlas in $\mathbb{R}^n$.*

(i) *If $0 \le s < 1/p$ and $1 \le q \le p$, then there exists $c > 0$ depending only on $\mathcal{A}, p, s$ and $q$ such that for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ and for all $u \in W_0^{1,p}(\Omega_2) \cap W^{1+s,p}(\Omega_2)$*

$$\|u\|_{L^q(\Omega_2 \setminus \Omega_1)} \le c \, d_{\mathcal{A},\bar{r}}(\Omega_1, \Omega_2)^{\bar{\gamma}} \|u\|_{W^{1+s,p}(\Omega_2)}, \tag{3.16}$$

*where*

$$\bar{r} = \left(1 + s + \frac{1}{q} - \frac{1}{p}\right)\mu \ \text{ with } \ \mu = \left(\frac{1}{q} - \frac{1}{p}\right)^{-1}, \quad \bar{\gamma} = 1 + s + \frac{1}{q} - \frac{1}{p}. \tag{3.17}$$

(ii) *If $s = 1/p$ and $1 \le q \le p$ then for any $\varepsilon \in (0,1)$ there exists $c_\varepsilon > 0$ depending only on $\varepsilon, \mathcal{A}, p, s$ and $q$ such that for all $\Omega_1, \Omega_2 \in C(\mathcal{A})$ and for all $u \in W_0^{1,p}(\Omega_2) \cap W^{1+s,p}(\Omega_2)$*

$$\|u\|_{L^q(\Omega_2 \setminus \Omega_1)} \le c \, d_{\mathcal{A},(1+\frac{1}{q}-\varepsilon)\mu}(\Omega_1, \Omega_2)^{1+\frac{1}{q}-\varepsilon} \|u\|_{W^{1+s,p}(\Omega_2)}, \tag{3.18}$$

*where $\mu = (1/q - 1/p)^{-1}$.*

(iii) *The estimate above holds also for $\varepsilon = 0$ if $s = 0$, $p = \infty$.*

**Proof.** The proof of this result goes much along the same lines as that of the preceding lemma. As in that lemma, by a suitable partition of unity, we may reduce to the case of domains $\Omega_i$, $i = 1, 2$ considered at the beginning of its proof. For $q_* = p/(1-sp)$ if $p < 1/s$ or $q_* = 1/\varepsilon$ if $p = 1/s$, or $q_* = \infty$ if $s = 0$, $p = \infty$, starting with (3.9), we obtain

$$\|u\|_{L^q(\Omega_2 \setminus \Omega_1)} \le \left\| (h_2(\bar{x}) - h_1(\bar{x}))^{\frac{1}{q}+\frac{1}{q'_*}} \left\| \frac{\partial u}{\partial x_n}(\bar{x}, \cdot) \right\|_{L^{q_*}(h_1(\bar{x}), h_2(\bar{x}))} \right\|_{L^q(W)}.$$

By Hölder's inequality

$$\|u\|_{L^q(\Omega_2\setminus\Omega_1)} \le \left\|(h_2(\bar{x})-h_1(\bar{x}))^{\frac{1}{q}+\frac{1}{q_*'}}\right\|_{L^\mu(W)} \left\|\left\|\frac{\partial u}{\partial x_n}(\bar{x},\cdot)\right\|_{L^{q_*}(h_1(\bar{x}),h_2(\bar{x}))}\right\|_{L^p(W)}$$

since $1/q = 1/\mu + 1/p$.

As in the previous lemma–that is, by using the one-dimensional continuous embedding $W^{s,p}(a_n, h_2(\bar{x})) \hookrightarrow L^{q_*}(a_n, h_2(\bar{x}))$ inequality, and the fact that the norm of the corresponding embedding operator is bounded above by a quantity $c > 0$ depending only on $p, s, \mathcal{A}$, see, e.g., [20]–we obtain

$$\|u\|_{L^q(\Omega_2\setminus\Omega_1)} \le c \left\|(h_2(\bar{x})-h_1(\bar{x}))\right\|_{L^{\bar{r}}(W)}^{\bar{\gamma}} \left\|\left\|\frac{\partial u}{\partial x_n}(\bar{x},\cdot)\right\|_{W^{s,p}(a_n,h_2(\bar{x}))}\right\|_{L^p(W)}$$

from which estimate (3.16) follows.

If $s = 1/p$ we apply the embedding theorem $W^{s,p}(a_n, h_2(\bar{x})) \hookrightarrow L^{q_*}(a_n, h_2(\bar{x}))$ with $q_* = 1/\varepsilon$ in order to derive (3.18). $\square$

In particular, under same assumptions and notation as in the lemma above, using (2.18), we obtain

$$\|u\|_{L^q(\Omega_2\setminus\Omega_1)} \le c\, d_{\mathcal{A},\infty}(\Omega_1, \Omega_2)^{\frac{1}{q}} d_{\mathcal{A},r}(\Omega_1, \Omega_2)^{\gamma} \|u\|_{W_p^{1+s}(\Omega_2)}, \tag{3.19}$$

where $r$, $\gamma$ are given by (3.3) if $p < 1/s$; $r = \mu\gamma$, $\gamma = 1-\varepsilon$, where $\mu = (1/q - 1/p)^{-1}$, $\varepsilon \in (0, 1)$, if $p = 1/s$; and $r = \infty$, $\gamma = 1$ if $s = 0$, $p = \infty$.

Now we are ready to give the

**Proof of Theorem 3.2.** It is not difficult to see that we may reduce to the case $\Omega \subset \Omega'$.

Let $\tilde{q} = \tilde{q}(\mathcal{F}, s)$, $\tilde{p} = \tilde{p}(\mathcal{F}, s)$ be as in Lemma 3.1 and $q_1$ as in Lemma 3.3. We take $\bar{q} = \bar{q}(\mathcal{F}, s) = \max\{\tilde{q}, q_1\}$ and $\bar{p} = \tilde{p}$ (in other words, the interval $(\bar{q}, \bar{p})$ is chosen in such a way that we may apply both (3.1) and (3.4) for $p, q \in (\bar{q}, \bar{p})$). Then by (3.4), (3.5), (3.1) we derive

$$\begin{aligned}\|u_\Omega - u_{\Omega'}\|_{L^q(\Omega)} &\le c\,\|(u_{\Omega'})|_{\partial\Omega}\|_{L^q(\partial\Omega)} \le c\, d_{\mathcal{A},r}(\Omega', \Omega)^{1+s-\frac{1}{p}} \|u_{\Omega'}\|_{W^{1+s,\,p}(\Omega')}\\ &\le c\, d_{\mathcal{A},r}(\Omega', \Omega)^{1+s-\frac{1}{p}} \|f\|_{W^{s-1,\,p}(D)},\end{aligned}$$

where constants $c > 0$ (not necessarily the same throughout the chain of inequalities) depend only on $\mathcal{F}, q, p, s$, which together with (3.19) yield the desired estimate (3.2). $\square$

Summing up, the derivation of stability estimates for solutions is based upon the successive application of two a priori estimates Lemma 3.3 and Lemma 3.1. Other variants of Lemma 3.1 lead to other variants of stability estimates. For example, in the framework if domains $\Omega$ with $C^1$-boundaries, it is known that if $f \in L^p(\Omega)$ with $p > n$, then the gradient of the corresponding solution $u_\Omega$ to (1.2) is bounded, more precisely,

$$\|\nabla u_\Omega\|_{L^\infty(\Omega)} \le c\,\|f\|_{L^p(\Omega)}$$

where $c > 0$ depends only on the $C^1$-character of $\Omega$, and $p$; see [24, Section 8.11]. Applying this fact together with Lemma 3.3, Lemma 3.4 and Lemma 3.5 (with $q = p = \infty$, $s = 0$) we derive the following

**Theorem 3.6.** *Let $\mathcal{F} = C^1_M(\mathcal{A})$, where $M > 0$ and $\mathcal{A}$ is an atlas, and let $p > n$. Then there exists $c = c(\mathcal{F}, p) > 0$ such that for all $\Omega, \Omega' \in \mathcal{F}$ and for all $f \in L^p(D)$, where $D$ is a fixed domain that contains all elements of $\mathcal{F}$, we have*

$$\|u_\Omega - u_{\Omega'}\|_{L^\infty(\Omega)} \le c\, d_{\mathcal{A},\infty}(\Omega, \Omega')\|f\|_{L^p(D)}.$$

*In addition, the estimate above holds for any $\Omega' \in C(\mathcal{A})$ (that is, no regularity assumption is needed on $\Omega'$ at all apart from the fact that the atlas distance should make sense; the solution $u_{\Omega'}$ should be understood in distributional sense) and for any $\Omega \in \mathcal{F}$ provided that $\Omega' \subset \Omega$. Even more, we can take $\Omega' \subset \Omega$ arbitrary above, provided that we replace $d_{\mathcal{A},\infty}(\Omega, \Omega')$ with the Hausdorff distance $d_{\mathcal{H}}(\Omega, \Omega')$.*

The proof of the second part of this theorem is identical to the proof of [1, Proposition 3.3.6].

### 3.2. Estimates for the gradients of solutions

Actually, it is possible a strengthening of Theorem 3.2 in the sense that we can put on the left-hand side of (3.2) a "stronger" norm, provided that we increment $\bar{q}$ if necessary, still keeping it $< 2$. Apart from being interesting in its own right, this observation plays a crucial role in proving the sharpness of the exponent $\gamma$, see Proposition 3.9 below.

Indeed, we begin by remarking that Lemma 3.3 continues to hold if $L^q(\Omega)$ is replaced by $B^{q,q^*}_{1/q}(\Omega)$, where $q^* = \max\{2, q\}$, at least for $q_2 < q < \infty$, for some $1 \le q_2 = q_2(\mathcal{F}) < 2$.

That is, if $\mathcal{F}$ is a family of domains as in Lemma 3.3, then there exists $1 \le q_2 = q_2(\mathcal{F}) < 2$ such that for all functions $u$ that satisfy the assumptions of Lemma 3.3 we have

$$\|u\|_{B^{q,q^*}_{1/q}(\Omega)} \le c\,\|u|_{\partial\Omega}\|_{L^q(\partial\Omega)}, \tag{3.20}$$

provided that $q_2 < q < \infty$. For the proof of this fact see [14, Theorem 5.15] for the case of the Laplace operator $S = -\Delta$, comments at the end of Section 8 in [16] for the general case of a variable coefficients operator $S$, and also [25, Remark V, p. 37].

Let $\mathcal{A}$ be an atlas, $M > 0$, and let $\mathcal{F} = C^{0,1}_M(\mathcal{A})$, and let $\bar{q} < 2 < \bar{p}$ be as in Theorem 3.2. Then, reasoning in a similar way as in the proof of Theorem 3.2, that is, using (3.20), (3.5), (3.1) and (3.7), we conclude that

$$\|u_\Omega - u_{\Omega'}\|_{B^{q,q^*}_{1/q}(\Omega\cap\Omega')} \le c\, d_{\mathcal{A},r}(\Omega, \Omega')^{1+s-\frac{1}{p}}\|f\|_{W^{s-1,p}(D)} \tag{3.21}$$

for all $\hat{q} = \max\{\bar{q}, q_2\} < q \le p < \bar{p}$. For $p = 2$, $s = 1/2$ we must replace $s$ with $s + \varepsilon$, for $\varepsilon > 0$ no matter how small, above

Next, we need the following Poincaré inequality kind of result.

**Lemma 3.7.** *Let* $\mathcal{A} = \left(d, \sigma, \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\right)$ *be an atlas in* $\mathbb{R}^n$*, and let* $1 \le q \le p$*,* $0 \le s \le 1$*. If* $p < 1/s$*, let*

$$\bar{r} = \left(s + \frac{1}{q} - \frac{1}{p}\right)\mu \ \text{ with } \ \mu = \left(\frac{1}{q} - \frac{1}{p}\right)^{-1}, \quad \text{and } \bar{\gamma} = s + \frac{1}{q} - \frac{1}{p}; \tag{3.22}$$

*if* $p = 1/s$*, for any* $\varepsilon \in (0,\ 1/q)$*, let* $\bar{\gamma} = 1/q - \varepsilon$*,* $\bar{r} = \bar{\gamma}\mu$*, where* $\mu$ *is defined as above; if* $s = 0$*,* $p = \infty$*, let* $\bar{\gamma} = 1/q$ *and* $r = 1$*. Then for some* $c > 0$

$$\|u\|_{L^q(\Omega_2 \setminus \Omega_1)} \le c\, d_{\mathcal{A},\bar{r}}(\Omega_1,\ \Omega_2)^{\bar{\gamma}} \|u\|_{W^{s,p}(\Omega_2)} \tag{3.23}$$

*for all* $\Omega_1$*,* $\Omega_2 \in C(\mathcal{A})$*, and for all* $u \in W^{s,\,p}(\Omega_2)$*.*

**Proof.** As in the proof of Lemma 3.4 or Lemma 3.5, by a suitable partition of unity, we may reduce to considering domains of the form $\Omega_k$, $k = 1, 2$ as in (3.8). Thus, using same notation as in the proof of the said lemma, for $u \in W^{s,\,p}(\Omega_2)$, $\bar{x} \in W$ such that $g_1(\bar{x}) < g_2(\bar{x})$ by Hólder's inequality

$$\|u(\bar{x},\cdot)\|_{L^q(g_1(\bar{x}),\,g_2(\bar{x}))} \le \left(g_2(\bar{x}) - g_1(\bar{x})\right)^{\frac{1}{q} - \frac{1}{q_*}} \|u(\bar{x},\cdot)\|_{L^{q_*}(g_1(\bar{x}),\,g_2(\bar{x}))}.$$

Next, we use the one-dimensional continuous embedding $W^{s,p}(a_n, g_2(\bar{x})) \hookrightarrow L^{q_*}(a_n, g_2(\bar{x}))$, where $q_* = p/(1 - sp)$ if $p < 1/s$, $q_* = 1/\varepsilon$ if $p = 1/s$, $q_* = \infty$ if $p = \infty$, $s = 0$, and the fact that the norm of the corresponding embedding operator is bounded above by a quantity $c > 0$ depending only on $p, s$ and $\mathcal{A}$, see, e.g., [20], in order to obtain

$$\|u(\bar{x},\cdot)\|_{L^q(g_1(\bar{x}),\,g_2(\bar{x}))} \le c\left(g_2(\bar{x}) - g_1(\bar{x})\right)^{\frac{1}{q} - \frac{1}{q_*}} \|u(\bar{x},\cdot)\|_{W^{s,\,p}(a_n,\,g_2(\bar{x}))}.$$

We derive (3.23) by integrating the $q$-th power of the inequality above over $\{\bar{x} \in W \ :\ g_1(\bar{x}) < g_2(\bar{x})\}$ and applying again Hólder's inequality since $1/\mu + 1/p = 1/q$. $\square$

Finally, we derive estimates for the variation of solutions of (1.2) in $W^{1,\,q}$-norm as a result of domain perturbation, via interpolation, in the following way. Let us assume that we are under the assumptions of Theorem 3.2, and let $\hat{q} < 2 < \bar{p}$ be such that (3.21) holds for $\hat{q} < q \le p < \bar{p}$. In addition, let $0 \le t < 1/q$ or $0 \le t \le 1/2$ if $q = 2$. Let $\theta \in [0,\ 1]$ be such that

$$\frac{\theta}{q} + (1 - \theta)(1 + t) = 1. \tag{3.24}$$

By real interpolation $\left(B_{1/q}^{q,\,q^*}(\Omega \cap \Omega'),\ B_{1+t}^{q,\,q}(\Omega \cap \Omega')\right)_{\theta,\,q_*} = B_1^{q,\,q_*}(\Omega \cap \Omega') \subset W^{1,\,q}(\Omega \cap \Omega')$, where $q_* = \min\{2,\ q\}$, applied to inequalities (3.21) and

$$\|u_\Omega\|_{B_{1+t}^{q,\,q}(\Omega\cap\Omega')} \le c\,\|f\|_{W^{t-1,\,q}(D)}$$

(if $q = 2$, $t = 1/2$, we must replace $t - 1$ in the right side above with $t - 1 + \varepsilon$ with $\varepsilon > 0$ no matter how small) which in tern is a consequence of the regularity estimate (3.1) with $p = q$, $s = t$, and, using also Lemma 3.7 applied to gradients of solutions, we obtain the following

**Theorem 3.8.** *Let $\mathcal{A}$ be an atlas, $M > 0$, and let $\mathcal{F} = C^{0,1}_M(\mathcal{A})$. Let $D$ be a bounded domain that contains all elements of $\mathcal{F}$. Let $1 \le p \le \infty$ and $0 \le s < 1/p$. Then there exist $\bar{q} = \bar{q}(\mathcal{F}, s) < 2$ and $\bar{p} = \bar{p}(\mathcal{F}, s) > 2$ such that for all $\bar{q} < q \le p < \bar{p}$, $0 \le t < 1/q$ and, for all $\Omega, \Omega' \in \mathcal{F}$, for some $c = c(\mathcal{F}, p, q, s, t) > 0$, we have*

$$\|u_\Omega - u_{\Omega'}\|_{W^{1,q}(D)} \le c \max\left\{d_{\mathcal{A},r}(\Omega, \Omega')^{\theta\gamma}, d_{\mathcal{A},\bar{r}}(\Omega', \Omega)^{\bar{\gamma}}\right\} \cdot \max\left\{\|f\|_{B^{p_\theta, q_*}_{s_\theta - 1}(D)}, \|f\|_{W^{s-1,p}(D)}\right\}, \tag{3.25}$$

*where $r$, $\gamma$ are defined as in Theorem 3.2, $\bar{r}$, $\bar{\gamma}$ as in Lemma 3.7, $\theta$ is determined by (3.24), $s_\theta = \theta s + (1-\theta)t$, $p_\theta = \theta p + (1-\theta)q$, $q_* = \min\{2, q\}$, and $f \in B^{p_\theta, q_*}_{s_\theta - 1}(D) \cap W^{s-1,p}(D)$; if $p = 2$, $s = 1/2$ and/or $q = 2$, $t = 1/2$ the estimate above continues to hold provided that $s$ and/or $t$ in the right-hand side of (3.25), and in the expression of $s_\theta$, are replaced by $s + \varepsilon$ and/or $t + \varepsilon$, respectively; (obviously, $u_\Omega$ denotes the solution to (1.2) extended to $D$ by setting it zero outside its domain of definition, clearly $u_\Omega \in W^{1,q}_0(D)$).*

### 3.3. Sharpness of exponent $\gamma$

We prove that exponent $\gamma$ in Theorem 3.2 is "sharp", in the sense that it cannot be replaced by any $\gamma' > \gamma$. Indeed, similarly to [2, Proposition 3.2], we note that an estimate like (3.2) induces extra regularity on solutions of the Dirichlet problem (1.2). But the amount of regularity of an arbitrary solution in the context of Lipschitz domains as prescribed by Lemma 3.1 is sharp (that is, one can always find a solution which is as regular as prescribed by that lemma but not more in the scale of fractional order Sobolev spaces) and this implies an upper bound on $\gamma$. More precisely, we can prove the following

**Proposition 3.9.** *Let $\mathcal{F} = C^{0,1}_M(\mathcal{A})$, where $\mathcal{A} = \left(d, \sigma, \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s\right)$ is an atlas, $M > 0$, and let $D$ be a bounded domain that contains all elements of $\mathcal{F}$. Assume that for fixed $1 \le p \le \infty$, $0 \le s \le 1$, $p < 1/s$, $r, \gamma \ge 0$, problem (1.2) has a solution $u_\Omega$ for all $\Omega \in \mathcal{F}$, and moreover, for all $1 \le q \le p$, estimate (3.2) holds for all $\Omega, \Omega' \in \mathcal{F}$. Then*

$$\gamma \le 1 + s - \frac{1}{p}. \tag{3.26}$$

**Proof.** First, it is easy to see that for a given atlas $\mathcal{A} = (d, \sigma, \{V_j\}_{j=1}^s, \{r_j\}_{j=1}^s)$ in $\mathbb{R}^n$ there exist $\delta, c \ge 0$ such that for all $\Omega \in C(\mathcal{A})$ and $|h| < \delta$, we have $\Omega + h \in C(\mathcal{A})$ and $d_{\mathcal{A},r}(\Omega + h, \Omega) \le c\,|h|$. Second, by estimate (3.2) and interpolation as above we derive that an estimate like (3.25) holds true. Third, as during the proof of [2, Proposition 3.2], we obtain that

$$\|u - u_h\|_{\Omega_{|h|}} \le c\,|h|^{\gamma'} \quad \forall |h| \le \delta,$$

where $\gamma' = \min\{\theta\gamma, \bar{\gamma}\}$, for $\theta$ determined by (3.24) for $t = s$, that is, $\theta = s/(1 + s - 1/q)$, $\bar{\gamma} = 1/q - 1/p - s$; here $u_h = u(\cdot - h)$ denotes the translation of $u$ by a vector $h \in \mathbb{R}^n$. But, after recalling that for a function $u \in L^q(\Omega)$ we have

$$u \in W^{1+s,q}(\Omega) \iff \int_{\mathbb{R}^n} \left( \frac{\|u - u_h\|_{W^{1,q}(\Omega_{|h|})}}{|h|^s} \right)^q \frac{dh}{|h|^n} < \infty,$$

we conclude that $u \in W^{1+s',q}$ for all $s' < \gamma'$. Since the regularity of an arbitrary solution given by Lemma 3.1 is optimal in the given range of parameters $s, p$ as explained in [14], at least for $S = -\Delta$, we must have $\gamma' \le s$, and taking $q$ as near to $p$ as necessary above, since $\bar{\gamma} > s$, we must have $\theta\gamma \le s$ and thus derive the claimed estimate (3.26). $\square$

### 3.4. Estimates for the gap between spectral subspaces and eigenfunctions

In order to apply the theory developed in Section 2 we need to give an operator formulation to problem (1.4). Let us use the assumptions and notation of Theorem 3.2. Then we define an operator

$$S_\Omega : D(S_\Omega) \subset L^q(\Omega) \to L^q(\Omega)$$

(we prefer to avoid denoting explicitly the dependence of $S_\Omega$ on $q$) where its domain of definition is of course

$$D(S_\Omega) = \{u \in W_0^{1,q}(\Omega) \ : \ \exists f \in L^q(\Omega) \text{ s.t. } u \text{ is the solution to (1.2)} \}$$

and, in that case, we set $S_\Omega u = f$.

A crucial fact for us is that for $q = 2$, $S_\Omega$ coincides with the operator arising from the standard weak/variational formulation of (1.2). Let us recall it briefly. Consider the nonnegative sesquilinear form

$$Q : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C} \tag{3.27}$$

in $L^2(\Omega)$ defined by

$$Q(u, v) = \int_\Omega \Big( \langle A(x)\nabla u, \nabla v\rangle + b(x)u(x)\overline{v(x)} \Big)\, dx \tag{3.28}$$

for all $u, v \in H_0^1(\Omega)$, where $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathbb{C}^n$. It is a well-known fact (see, e.g., [26, Chapter 6]) that

$$S_\Omega : D(S_\Omega) \subset L^2(\Omega) \to L^2(\Omega)$$

is the unique positive selfadjoint operator that satisfies: $u \in D(S_\Omega)$ if and only if $u \in H_0^1(\Omega)$ and there exists (a unique) $f \in L^2(\Omega)$ such that

$$Q(u, v) = < f, v >_{L^2(\Omega)}$$

for all $v \in H_0^1(\Omega)$; in such a case $S_\Omega u = f$. Of course, to study the eigenvalue problem (1.4) in this paper means to study the spectrum and the eigenfunctions of operator $S_\Omega$. Since $H_0^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ (because $\Omega$ is bounded) the operator $S_\Omega$ has compact resolvent and therefore its spectrum is discrete. Actually, $S_\Omega$ as an operator in $L^q(\Omega)$ has discrete spectrum for all $q$ varying in the allowed range $]\bar{q},\, \bar{p}[$. Moreover, operators $S_\Omega$ have the same sequences of eigenvalues $\{\lambda_k[\Omega]\}_{k=1}^\infty$, arranged in ascending order, and eigenfunctions $\{\varphi_k[\Omega]\}_{k=1}^\infty$ for all $q \in ]\bar{q},\, \bar{p}[$.

Another issue that we need to tackle in order to be able to apply the results of the previous section is to find a way to see the operators $S_\Omega$, $\Omega \in \mathcal{F}$, as operators acting in a same normed space. We achieve this in a rather straightforward manner, that is, by "extending by zero". We are going to see operators $S_\Omega$ as operators acting in the space $L^q(D)$, where $D$ is, as usual, a fixed or universal domain that contains all elements of $\mathcal{F}$. Let us note that we have a natural identification

$$L^q(D) \equiv L^q(\Omega) \oplus L^q(D \setminus \Omega)$$

(the reader can guess what the identification map is; if $q = 2$, the direct sum above is an orthogonal direct sum). Then, we define

$$S_\Omega : D(S_\Omega) \oplus L^q(D \setminus \Omega) \subset L^q(D) \to L^q(D)$$

to be zero on $L^q(D \setminus \Omega)$ and then extend it by linearity to its new domain of definition $D(S_\Omega) \oplus L^q(D \setminus \Omega)$. (We are continuing to denote these operators with the same symbol $S_\Omega$, which is a slight abuse of notation.)

Of course, $S_\Omega$ for $q = 2$ belongs to the class of operators considered in Subsect. 2.2: in particular, its range is $L^2(\Omega)$ (seen as a subspace of $L^2(D)$ by extending its elements to $D$ by zero) and its restriction to $L^2(\Omega)$ has compact resolvent.

Let $k,\, m \in \mathbb{N}$ be such that inequalities (1.6) hold, then inequalities (2.5) for $T = S_\Omega$ (as an operator acting in $L^2(D)$) also hold, and

$$N_{k,\,m}[\Omega] = N_{k,\,m}[S_\Omega]$$

(up to the usual identification of course, where $N_{k,\,m}[\Omega]$ is seen as a subspace of $L^2(D)$, by trivially extending its elements by zero outside $\Omega$).

Since for all $f \in N_{k,\,m}[\Omega]$ we have $\|f\|_{L^2(D)} \le c\,\|f\|_{L^q(D)}$ for some $c = c(\Omega,\, q,\, k,\, m)$, (for $q \ge 2$ we can chose $c$ independent of $\Omega$; the story is different when $q < 2$ since we have to appeal to the fact that $N_{k,\,m}[\Omega]$ is finite-dimensional in order to find a $c > 0$, which inevitably depends also on $\Omega$, $k$, and $m$, such that the said inequality holds) we can use bootstrapping of the regularity for eigenfunctions (for $f$ is the sum of a finite number of eigenfunctions) in order to deduce that $\|f\|_{W^{s-1,\,p}(D)} \le c\,\|f\|_{L^q(D)}$ for some $c = c(\mathcal{F},\, p,\, q,\, s,\, \Omega,\, k,\, m) > 0$, which in turn by Theorem 3.2 implies that

$$\|(S_\Omega^{-1} - S_{\Omega'}^{-1})|_{N_{k,\,m}[\Omega]}\|_{L^q(D)} \le c\, d_{\mathcal{A},\,r}(\Omega,\, \Omega')^\gamma, \tag{3.29}$$

for some $c = c(\mathcal{F},\, p,\, q,\, s,\, \Omega,\, k,\, m) > 0$, where $r$, $\gamma$ are given by (3.3). Actually, for $q \ge 2$ we can chose $c$ above independent of $\Omega$ (as far as $\Omega \in \mathcal{F}$), $k$ and $m$.

Estimate (3.29) together with known estimates about eigenvalues [13, Theorem 5.1] allow us to apply Theorem 2.5 or Theorem 2.8 with $\mathcal{H} = L^2(D)$, $\mathcal{X} = L^q(D)$ (for $q \ne 2$) $\mathcal{E} = L^1(D)$, $T = S_\Omega$ for $q = 2$ and $\bar{T} = S_\Omega$ for $q \ne 2$ (by bootstrapping of the regularity, it follows that $T$ and $\bar{T}$ have the same generalized eigenfunctions) and thus deduce the following

**Theorem 3.10.** *Let $\mathcal{A}$ be an atlas in $\mathbb{R}^n$, $M > 0$ and let $\mathcal{F} = C_M^{0,1}(\mathcal{A})$. Let $1 \le p \le \infty$, $0 \le s < 1/p$, or $s = 1/2$ if $p = 2$, and $\bar{q} < q \le p < \bar{p}$, where $\bar{q}$, $\bar{p}$ are as in Theorem 3.2.*[5] *Let $\Omega \in C_M^{0,1}(\mathcal{A})$ and let $k, m \in \mathbb{N}$ be such that inequalities (1.6) hold. Then there exist $c = c(\mathcal{F}, p, q, s, \Omega, k, m) > 0$, $\delta = \delta(\mathcal{F}, p, q, s, \Omega, k, m) > 0$ such that for all $\Omega' \in C_M^{0,1}(\mathcal{A})$*

$$\hat{\delta}_{L^q(D)}(N_{k,m}[\Omega], N_{k,m}[\Omega']) \le c\, d_{\mathcal{A},r}(\Omega, \Omega')^{\gamma}, \tag{3.30}$$

*where $r$ and $\gamma$ are given by (3.3), whenever $d_{\mathcal{A},r}(\Omega, \Omega') \le \delta$.*

*For all $0 < \varepsilon < 1$ the result holds also for $p = 2$, $s = 1/2$ with $r = \infty$, $\gamma = 1 - \varepsilon$. (In this case the constant $c$ above depends also on $\varepsilon$.)*

*If $M$ is small enough or if $\mathcal{F} = C_M^1(\mathcal{A})$ the result holds with $\bar{q} = 1$ and $\bar{p} = \infty$.*

*If $\mathcal{F} = C_M^2(\mathcal{A})$ the result holds also for $p = \infty$, $s = 0$ and any $1 \le q \le p$ (in this case $\gamma = 1$, $r = q$).*

*If $\mathcal{F} = C_M^1(\mathcal{A})$ the result holds also for $q = p = \infty$ and $s = 0$ (which imply $r = \infty$, $\gamma = 1$). Even more, in this case the result holds also for all inner perturbations $\Omega'$ of $\Omega$, that is, $\Omega' \subset \Omega$, without requiring any regularity assumption on $\Omega'$, provided that $d_{\mathcal{A},\infty}(\Omega, \Omega')$ is replaced by the Hausdorff distance $d_{\mathcal{H}}(\Omega, \Omega')$.*

It is implicit in the statement above that under the said conditions inequalities (1.6) hold also for $\Omega'$ instead of $\Omega$ and thus $N_{k,m}[\Omega']$ is well defined. Of course, the last two assertions in the preceding theorem are obtained by applying the previous considerations together with Theorem 3.6.

**Remark 3.11.** *Going through the proof of Theorem 3.10, it is possible to give a more explicit description of the dependence of the constants $c$ and $\delta$ of that theorem on the domain $\Omega$. Indeed, under the assumptions of Theorem 3.10 and using the same notation, it is not difficult to see that we can take, for $q \ge 2$,*

$$c(\mathcal{F}, p, q, s, \Omega, k, m) = \frac{c_0(\mathcal{F}, p, q, s, m)}{\min\{\lambda_k[\Omega] - \lambda_{k-1}[\Omega], \lambda_{k+m}[\Omega] - \lambda_{k+m-1}[\Omega]\}}$$

*and, for $q < 2$,*

$$c(\mathcal{F}, p, q, s, \Omega, k, m) = \frac{c_0(\mathcal{F}, p, q, s, m)}{\min\{\lambda_k[\Omega] - \lambda_{k-1}[\Omega], \lambda_{k+m}[\Omega] - \lambda_{k+m-1}[\Omega]\}} \sup_{\substack{f \in N_{k,m}[\Omega] \\ f \ne 0}} \frac{\|f\|_{L^2(\Omega)}}{\|f\|_{L^q(\Omega)}}$$

*for some $c_0(\mathcal{F}, p, q, s, m) > 0$. As for $\delta$, analogous formulas hold (which we do not write to save space), with the difference that the "spectral gap", that is, the quantity $\min\{\lambda_k[\Omega] - \lambda_{k-1}[\Omega], \lambda_{k+m}[\Omega] - \lambda_{k+m-1}[\Omega]\}$ stands in the numerator, and, for $q < 2$, the supremum passes to the denominator; after all, we may take $\delta = 1/c$, enlarging $c_0$ if necessary.*

In a similar way, by applying Theorem 2.5 or Theorem 2.8, to suitable operators (the reader may identify these operators easily) we have the following result.

[5] In particular, recall that if $s = 0$, then $\bar{p} = \bar{p}(\mathcal{F}, 0) > 3$.

**Theorem 3.12.** *Let $\mathcal{F}$, $M$, $q$, $p$, $s$, $t$ be as in Theorem 3.8. Let $\Omega \in \mathcal{F}$ and $k, m \in \mathbb{N}$ be such that inequalities (1.6) hold. Them, for suitable $c = c(\mathcal{F}, p, q, s, t, \Omega, k, m) > 0$, $\delta = \delta(\mathcal{F}, p, q, s, t, \Omega, k, m) > 0$,*

$$\hat{\delta}_{W^{1,q}(D)}(N_{k,m}[\Omega], N_{k,m}[\Omega']) \le c \max\left\{d_{\mathcal{A},r}(\Omega, \Omega')^{\theta\gamma}, d_{\mathcal{A},\bar{r}}(\Omega', \Omega)^{\bar{\gamma}}\right\}, \tag{3.31}$$

*where $r$, $\gamma$ are given by (3.3), $\bar{r}$, $\bar{\gamma}$ by (3.17), and $\theta$ is determined by (3.24), for all $\Omega' \in \mathcal{F}$ provided that* $\max\left\{d_{\mathcal{A},r}(\Omega, \Omega')^{\theta\gamma}, d_{\mathcal{A},\bar{r}}(\Omega', \Omega)^{\bar{\gamma}}\right\} \le \delta$.

These stability estimates for spectral subspaces yield also certain stability estimates for eigenfunctions. By the definition of the gap follows that any eigenfunction $\psi'$ of $S_{\Omega'}$ is "close" in $L^q$ or $W^{1,q}$-norm, for suitable "$q$"s, to some eigenfunction $\psi$ of $S_\Omega$. For example, under the assumptions of Theorem 3.10, it follows that

$$\|\psi' - \psi\|_{L^q(D)} \le c\, d_{\mathcal{A},r}(\Omega, \Omega')^\gamma \|\psi'\|_{L^q(\Omega')}.$$

In a Hilbert space context, by appealing to Lemma 2.3 we can approximate by "preserving the orthogonality relation", that is, each element of any finite sequence of eigenfunctions of $S_{\Omega'}$, orthonormal in $L^2(\Omega')$ (or in $H^1(\Omega')$) is "close" in $L^2$-norm (or in $H^1$-norm) to some element of some finite sequence of orthonormal eigenfunctions of $S_\Omega$ in $L^2(\Omega)$ (or in $H^1(\Omega)$). More precisely, by using e.g., Theorem 3.10, we obtain:

**Theorem 3.13.** *Let $\mathcal{F}$, $M$, $q$, $p$, $s$, $r$ and $\gamma$ be as in Theorem 3.2. Assume in addition that $q = 2$. Let $\Omega \in \mathcal{F}$, and let $k, m \in \mathbb{N}$ be such that inequalities (1.6) hold. Then there exist $c, \delta > 0$ such that the following holds: if $\Omega' \in \mathcal{F}$ is such that $d_{\mathcal{A},r}(\Omega, \Omega') \le \delta$, and $\psi_k[\Omega'], \dots, \psi_{k+m-1}[\Omega']$ is an orthonormal set in $L^2(\Omega')$ of eigenfunctions of $S_{\Omega'}$ corresponding to the eigenvalues $\lambda_k[\Omega'] \le \dots \le \lambda_{k+m-1}[\Omega']$, then there exists an orthonormal set in $L^2(\Omega)$ of eigenfunctions $\psi_k[\Omega], \dots, \psi_{k+m-1}[\Omega]$ of $S_\Omega$ corresponding to the eigenvalues $\lambda_k[\Omega] \le \dots \le \lambda_{k+m-1}[\Omega]$ of $S_\Omega$ such that*

$$\|\psi_{k+i}[\Omega] - \psi_{k+i}[\Omega']\|_{L^2(D)} \le c\, d_{\mathcal{A},r}(\Omega, \Omega')^\gamma$$

*for all $i = 0, \dots, m-1$.*

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