# Vector Analysis

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# Outline

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# What is a vector?

- An **ordered** system of numbers.
- A geometrical transformation called translation
- An oriented segment, an “arrow”., that is, “something” which has a length, a direction and a verse.

Emphasis: **ordered**

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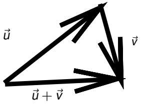
# Basic operations with vectors: addition

Existence makes sense in terms of others, in a context. Vectors are entities with which one can make certain “algebraic operations” and produce other similar entities.

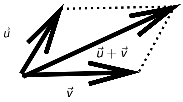
Given vectors in $\mathbb{R}^3$ $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k} \equiv \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k} \equiv \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$

Addition:

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{pmatrix}$$




Triangle rule




Parallelogram rule

## Multiplication of a vector by a scalar

Given also a **scalar** $\lambda \in \mathbb{R}$

$$\lambda\,\vec{v} = \begin{pmatrix} \lambda\, v_1 \\ \lambda\, v_2 \\ \lambda\, v_3 \end{pmatrix}$$

**Properties:**

- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{u} + \vec{w})$
- $\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$
- $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$
- $1\,\vec{v} = \vec{v}$,

etc.

# Basic operations with vectors: Multiplication by a scalar

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# Vector spaces

Mathematicians like to generalise or extrapolate:
Scalars are entities with which one can perform certain binary operations called "addition" and "multiplication" satisfying certain rules called axioms. One speaks of a "field of scalars". $\mathbb{R}, \mathbb{C}, \mathbb{Q}$ are examples "fields of scalars" but $\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_p$, $p$ a prime integer, are examples as well.

Vectors: over a field of scalars are certain entities with which one can perform certain operations called "addition of vectors" and "multiplication of vectors by scalars" producing other similar entities obeying certain axioms. The entire collection of such entities is called a Vector space over a field of scalars. Examples: $\mathbb{R}^n$, $M_{m,n}(\mathbb{R})$, $\mathbb{R}[x]$, polynomials in with coefficients in $\mathbb{R}$.

# Products

Scalar product or "dot product":

$$\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3$$

where $\theta$ is angle between $u$ and $v$

Vector product or "cross product"

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\vec{u} \times \vec{v}$ is the unique vector in $\mathbb{R}^3$ which
1. is orthogonal to both $\vec{u}$ and $\vec{v}$,
2. whose verse is determined by the "right hand rule",
3. $|\vec{u} \times \vec{v}| = |\vec{u}|\,|\vec{v}| \sin\theta$.

# Triple products

Scalar triple product:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

$|\vec{u}\dot{}(\vec{v} \times \vec{w})|$ is the signed volume of parallelepiped spanned by $\vec{u}$, $\vec{v}$, $\vec{w}$.

Vector triple product:

$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Mnemonic: if $\vec{u} = a$, $\vec{v} = b$, $\vec{w} = c$, the result is "BAC-CAB", that is, $a\,(b \cdot c) - c\,(a\dot{b})$. (Lagrange's identity).

# Standard and less standard facts I

Scalar product is bilinear, commutative or symmetric and positive definite. Cauchy-Schwarz's inequality $|\vec{u}\cdot\vec{v}| \leq |\vec{u}|\,|\vec{v}|$ with equality if and only if $u \parallel \vec{v}$. $\vec{u}\cdot\vec{v} = 0$ iff $\vec{u} \perp \vec{v}$.

Vector product is bilinear and antisymmetric: $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$. It is not associative but fulfills Jacobi's identity

$$(\vec{u}\times\vec{v})\times\vec{w} + (\vec{v}\times\vec{w})\times\vec{u} + (\vec{w}\times\vec{u})\times\vec{v} = \vec{0}$$

- $|\vec{u}\times\vec{v}|^2 = \begin{vmatrix} \vec{u}\cdot\vec{u} & \vec{u}\cdot\vec{v} \\ \vec{v}\cdot\vec{u} & \vec{v}\cdot\vec{u} \end{vmatrix} = |\vec{u}|^2|\vec{v}|^2 - (\vec{u}\cdot\vec{v})^2$ (Lagrange's identity). Length of $\vec{u}\times\vec{v}$ = Gram's determinant of $\vec{u}$ and $\vec{v}$.
- Particular case of $(\vec{u}\times\vec{v})\cdot(\vec{w}\times\vec{z}) = \begin{vmatrix} \vec{u}\cdot\vec{w} & \vec{u}\cdot\vec{z} \\ \vec{v}\cdot\vec{w} & \vec{v}\cdot\vec{w} \end{vmatrix} = (\vec{u}\cdot\vec{w})(\vec{v}\cdot\vec{w}) - (\vec{v}\cdot\vec{w})(\vec{u}\cdot\vec{z})$.
- $(\vec{u}\times\vec{v})\times(\vec{w}\times\vec{u}) = (\vec{u}\cdot(\vec{v}\times\vec{w}))u$.

# Standard and less standard facts II

Triple products are trilinear. Scalar triple product is alternating and invariant for cyclings and swapping of algebraic operations (operands being kept fixed):

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$$
$$\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$$
$$\vec{u}\cdot(\vec{v}\times\vec{w}) = -\vec{v}\cdot(\vec{u}\times\vec{w}).$$
$$((\vec{a}\times\vec{b})\cdot\vec{c})((\vec{u}\times\vec{v})\cdot\vec{w}) = \det\left[\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix}\begin{pmatrix}\vec{u} & \vec{v} & \vec{w}\end{pmatrix}\right]$$

If $T$ is an isometry

$$T\vec{u}\cdot(T\vec{v}\times T\vec{w}) = \vec{u}\cdot(v\times w)$$

if $T$ is a proper, and

$$T\vec{u}\cdot(T\vec{v}\times T\vec{w}) = -\vec{u}\cdot(v\times w)$$

if $T$ is improper.

# Outline

# Vector-valued functions I

These are maps of one or more variables whose range is a set of multidimensional vectors. Dimensions of domain and codomain could be different.

Vector functions of a single variable $t$, usually interpreted as time:

$$\vec{r}\colon I \subset \mathbb{R} \to \mathbb{R}^n, \qquad t \mapsto \vec{r}(t).$$

Could represent the "time law" of a material particle in motion, that is, $\vec{r}(t)$ is position of the particle at time $t$. Derivative:

$$\frac{d\vec{r}(t)}{dt} = \vec{r}'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

If $\vec{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}$, then $\vec{r}'(t) = \begin{pmatrix} r_1'(t) \\ r_2'(t) \\ r_3'(t) \end{pmatrix}$. If $t \mapsto \vec{r}(t)$ is the motion law of a

# Vector-valued functions II

particle

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt}, \qquad \vec{a}(t) = \frac{d^2\vec{r}(t)}{dt^2}$$

are its velocity and acceleration.

**Theorem (Leibniz's product rule of differentiation)**

*If $* : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a bilinear map and $\vec{r}_1 : I \subset \mathbb{R} \to \mathbb{R}^n$ and $\vec{r}_2 : I \subset \mathbb{R} \to \mathbb{R}^m$ are derivable at some $t_0$, then $\vec{r}_1 * \vec{r}_2 : I \to \mathbb{R}^p$, $t \mapsto \vec{r}_1(t) * \vec{r}_2(t)$ is derivable at $t_0$ and*

$$(\vec{r}_1 * \vec{r}_2)'(t_0) = \vec{r}_1'(t_0) * \vec{r}_2(t_0) + \vec{r}_1(t_0) * \vec{r}_2'(t_0).$$

**Exercise:** In circular motion velocity and position (radius vector from the origin) are perpendicular at each time.

**Exercise:** A particle under the action of a central force moves in a plane perpendicular to the constant angular momentum: $\vec{r}(t)\vec{a}(t) = \vec{0}$.

# Scalar fields and vector fields, differential operators and integral theorems I

Scalar field:
Assignment of a scalar in each point of a region of space.
Examples: Density, temperature, pressure.

Vector field:
Assignment of a vector in each point of a region of space.
Examples: gravitational field, field of velocities of flowing fluid, electric field, magnetic field

Gradient: Given a scalar field $f\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}$

$$\mathrm{grad}(f) = \nabla f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$$

Maps a scalar field into a vector field. Measures the magnitude and direction of change of the scalar field.

# Scalar fields and vector fields, differential operators and integral theorems II

Gradient Theorem For a trajectory $\gamma\colon [a,b] \to \Omega$ and a scalar field $V\colon \Omega \to \mathbb{R}$,

$$\int_\gamma \nabla V \cdot d\vec{\ell} = V(\gamma(b)) - V(\gamma(a)).$$

A vector field $\vec{F}$ is called conservative if $\vec{F} = \nabla V$ for some scalar field $V$. In Mechanics, when $F$ is seen as a force field, $-V$ is called its potential energy. Work done by a conservative force on a particle does not depend on trajectory but only on starting point and end point. In particular work on closed loops is zero.

Given $\mathbf{F}\colon \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$
Divergence:

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

# Scalar fields and vector fields, differential operators and integral theorems III

It is a scalar field and measures the density of the flow of a vector field emerging or accumulating at a point. It quantifies a point as a source or sink.

Divergence theorem (or Gauss' theorem)

$$\iiint_{\Omega} \nabla \vec{F}\, dV = \oiint_{\partial\Omega} \vec{F} \cdot \vec{n}\, dS.$$

curl:

$$\mathrm{curl}(\vec{F}) \; = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

It is a vector field and measures the tendency of a vector field to rotate about a point.

# Scalar fields and vector fields, differential operators and integral theorems IV

Stokes' theorem: For a surface $S$ in $\mathbb{R}^3$ with boundary $\partial S$ a closed curve and a vector field $\vec{F} \to \Omega \to \mathbb{R}^3$ such that $S \cup \partial S \subset \Omega$ one has

$$\iint_S \nabla \vec{F} \cdot \vec{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{\ell}\,.$$

Laplacian is

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \sum_{i=1}^{3} \frac{\partial^2 f}{\partial x_i^2}.$$

Measures the difference between the value of a scalar field at a point and its average on infinitesimal balls centered at that point. Maps scalar fields into scalar fields.

# Scalar fields and vector fields, differential operators and integral theorems V

Vector Laplacian: For a vector field $\vec{F}: \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$ vector laplaciaon is

$$\nabla^2 \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla \times (\nabla \times \vec{F})$$

Measures the difference between the value of the vector field with its average on infinitesimal balls. Maps vector fields into vector fields.

# Maxwell's equations

| Name | Integral equation | Differential equation |
|---|---|---|
| Gauss's law | $\oiint_{\partial\Omega} \vec{E} \cdot d\vec{S} = \frac{1}{\varepsilon_0} \iiint_{\Omega} \rho \, dV$ | $\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ |
| Gauss's law of magnetism | $\oiint_{\partial\Omega} \vec{B} \cdot d\vec{S} = 0$ | $\nabla \cdot \vec{B} = 0$ |
| Faraday's law of induction | $\oint_{\partial\Sigma} \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \iint_{\Sigma} \vec{B} \cdot d\vec{S}$ | $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ |
| Ampère's circuital law (with Maxwell's addition) | $\oint_{\partial\Sigma} \vec{B} \cdot d\vec{\ell} = \mu_0 \left( \iint_{\Sigma} \vec{J} \cdot d\vec{S} + \varepsilon_0 \frac{d}{dt} \iint_{\Sigma} \vec{E} \cdot d\vec{S} \right)$ | $\nabla \times \vec{B} = \mu_0 \left( \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$ |

Many Thanks for Your Attention!