Stability estimates for solutions and eigenfunctions of second-order elliptic Dirichlet BVPs under domain perturbation

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Stability problem for solutions

Given a second-order unif. elliptic divergent form differential operator

$$Su = -\mathrm{div}(A(x)\nabla u) + b(x)u \tag{1}$$

with loc. Lip. coeffic.: $A(x) \geq \theta I_n$,, $\theta > 0$, $b(x) \geq 0$ $\forall x$, and a family $\mathcal{F}$ of bounded domains in $\mathbb{R}^n$, consider in any pair of such domains Ω, $\Omega' \in \mathcal{F}$ the problems

$$\begin{cases} Su = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}, \qquad \begin{cases} Su' = f & \text{in } \Omega', \\ u' = 0 & \text{on } \partial\Omega' \end{cases}, \tag{DP}$$

where $f \in \mathcal{W}(D)$, a function space of functions defined on D, a fixed universal domain that contains all elements of $\mathcal{F}$. The aim is to establish estimates of the type

$$\|u - u'\|_{\mathcal{Z}(D)} \leq c \cdot d(\Omega, \Omega')^{\gamma} \|f\|_{\mathcal{W}(D)}, \tag{2}$$

where finding $\mathcal{F}$, $d(\cdot,\cdot)$, D, $\mathcal{X}(D)$, $\mathcal{W}(D)$, $0 < \gamma \leq 1$ is part of the problem.

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Stability problem for eigenfunctions

Consider the eigenvalue problem

$$\begin{cases} Su = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{3}$$

Spectral subspaces: Let

- $$\lambda_1[\Omega] \leq \cdots \leq \lambda_k[\Omega] \leq \lambda_{k+1}[\Omega] \leq \cdots \to \infty$$

  be the eigenvalues of S_Ω listed in ascending order, each repeated as many times as its multiplicity
- $\{\varphi_k[\Omega]\}_{k\in\mathbb{N}}$ an orthonormal basis of eigenfunctions in $L^2(\Omega)$.

If $k, m \in \mathbb{N}$ are such that

$$\lambda_{k-1}[\Omega] < \lambda_k[\Omega], \qquad \lambda_{k+m-1}[\Omega] < \lambda_{k+m}[\Omega], \tag{4}$$

we define

$$N_{k,m}[\Omega] = \operatorname{span}\{\varphi_k[\Omega], \cdots, \varphi_{k+m-1}[\Omega]\}. \tag{5}$$

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Stability problem for eigenfunctions: prototypical result

Find or exhibit

- $\mathcal{F}$, a family of domains,
- $d(\cdot, \cdot)$ a distance on $\mathcal{F}$,
- $\mathcal{Z}$ normed space of functions D

$$N_{k,m}[\Omega] \subset \mathcal{Z} \qquad \forall \Omega \in \mathcal{F};$$

- $0 < \gamma \, (\leq 1)$,

such that theorems of the following kind may be formulated

Theorem (prototypical result)

Given $\Omega \in \mathcal{F}$ and $k,\ m \in \mathbb{N}$ s.t. ineq. (4) hold, then $\exists\ \delta,\ c > 0$ s.t.

$$\hat{\delta}(N_{k,m}[\Omega],\ N_{k,m}[\Omega']) \leq c \cdot d(\Omega,\ \Omega')^{\gamma}. \tag{6}$$

for all $\Omega' \in \mathcal{F}$ provided that

$$d(\Omega,\ \Omega') \leq \delta.$$

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Gap between subspaces

Let $\mathcal{M}$, $\mathcal{N}$ be linear subspaces of a normed space $\mathcal{Z}$. Consider the inequality

$$\operatorname{dist}(u, \mathcal{N}) \leq c\|u\| \quad \forall u \in \mathcal{M},$$

which holds for some $0 \leq c \leq 1$. The optimal **c** above is called the *gap of* $\mathcal{M}$ to $\mathcal{N}$. One calls

$$\hat{\delta}(\mathcal{M}, \mathcal{N}) = \hat{\delta}_{\mathcal{Z}}(\mathcal{M}, \mathcal{N}) = \max\{\delta(\mathcal{M}, \mathcal{N}), \delta(\mathcal{N}, \mathcal{M})\}.$$

the *gap between* $\mathcal{M}$ and $\mathcal{N}$. If, in addition, $\mathcal{Z}$ is a Hilbert space, then

$$\hat{\delta}(\mathcal{M}, \mathcal{N}) = \|P_{\overline{\mathcal{M}}} - P_{\overline{\mathcal{N}}}\|,$$

where $P_{\overline{\mathcal{M}}}$, $P_{\overline{\mathcal{N}}}$ are the orthogonal projectors on $\overline{\mathcal{M}}$ ed $\overline{\mathcal{N}}$, respectively. In a Hilbert space $\mathcal{Z}$ for all $e_1, e_2 \in \mathcal{Z}$

$$\hat{\delta}(< e_1 >, < e_2 >) = \sqrt{1 - \left(\frac{|(e_1, e_2)|}{\|e_1\|\|e_2\|}\right)^2}.$$

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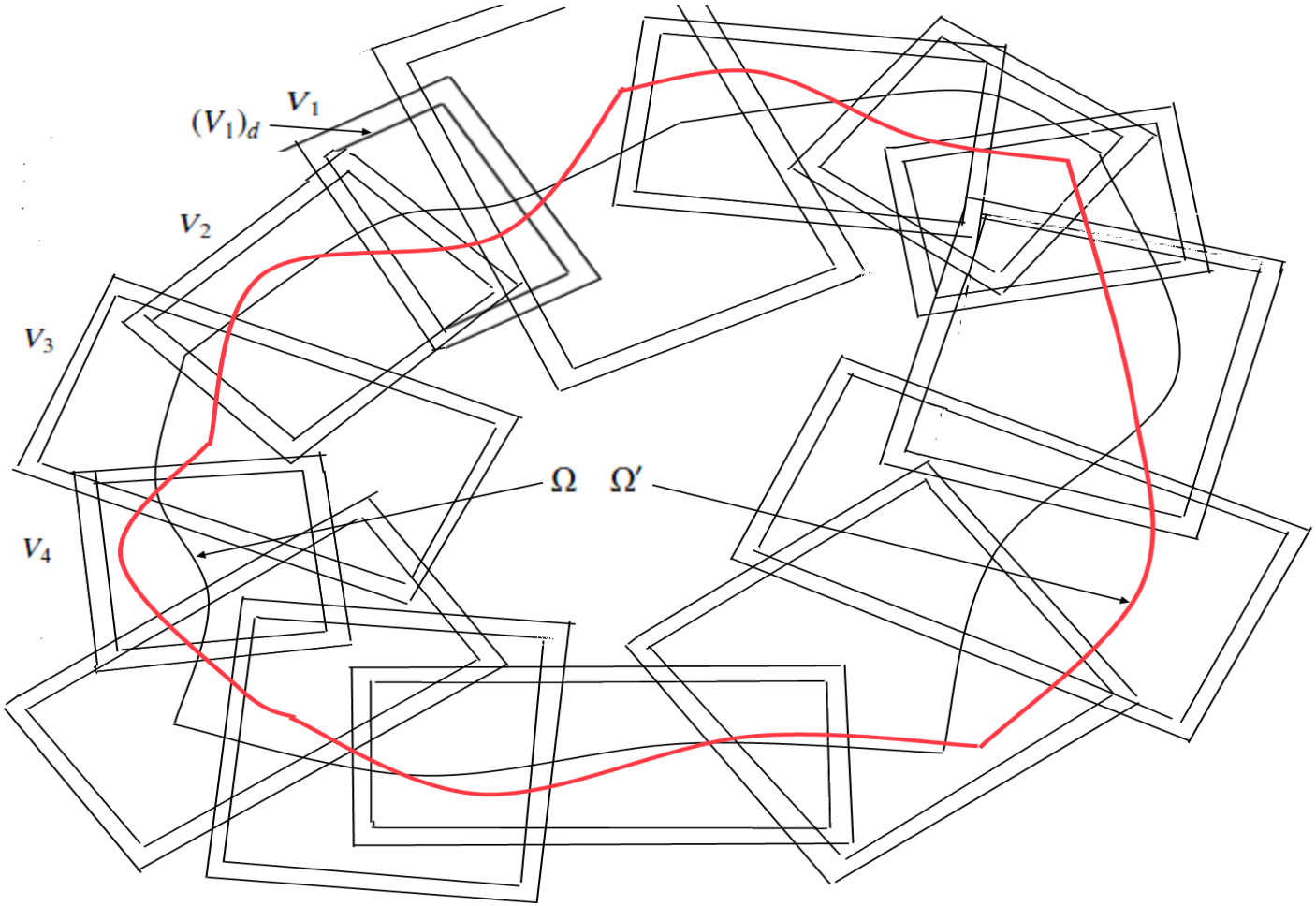
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An atlas describing the boundaries of two domains



An atlas $\mathcal{A} = (d > 0,\ \sigma \in \mathbb{N},\ \{V_j\}_{j=1}^{\sigma}$ cuboids, $\{r_j\}_{j=1}^{\sigma}$ rotations)

Atlas distances

$\Omega \in C(\mathcal{A})$ means that

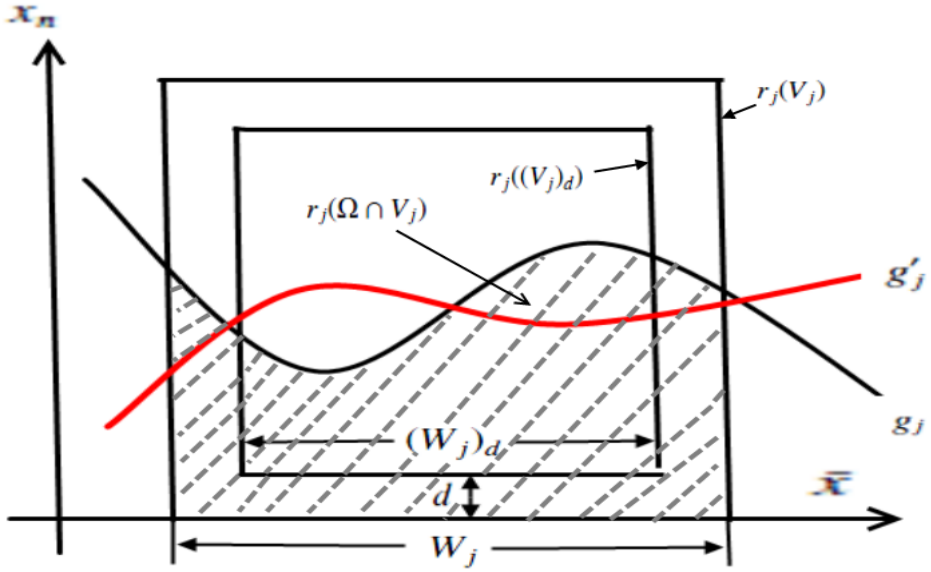
(i) $\partial\Omega \subset \bigcup_{j=1}^{\sigma} (V_j)_d$;

(ii) $(V_j)_d \cap \partial\Omega \neq \emptyset$ for $j = 1, \dots \sigma$.

and that $r_j(\Omega \cap V_j) =$

$\{x \in \mathbb{R}^n : \ a_{nj} < x_n < g_j(\bar{x}), \ \bar{x} \in W_j\}$

for continuous functions g_j.



Similarly $C^1(\mathcal{A})$, $C^2(\mathcal{A})$, $C^{0,1}(\mathcal{A})$, $C^{0,\alpha}(\mathcal{A})$ etc...and, if the norms of g_j-s are bounded by $M > 0$, $C^1_M(\mathcal{A})$, $C^2_M(\mathcal{A})$, $C^{0,1}_M(\mathcal{A})$, $C^{0,\alpha}_M(\mathcal{A})$ etc... Let $\Omega, \ \Omega' \in C(\mathcal{A})$, $1 \leq r \leq \infty$, we define the atlas distances

$$d_{\mathcal{A},r}(\Omega, \Omega') = \left(\sum_{j=1}^{\sigma} \|g_j - g'_j\|^r_{L^r(W_j)} \right)^{1/r}.$$

Fix $M > 0$ and an atlas $\mathcal{A}$. Let $\mathcal{F} = C^{0,1}_M(\mathcal{A})$.

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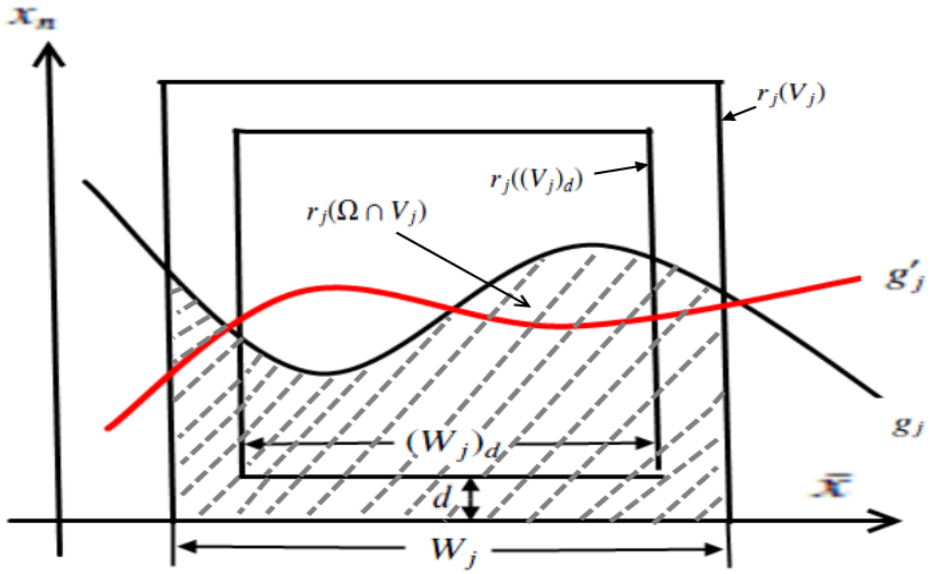
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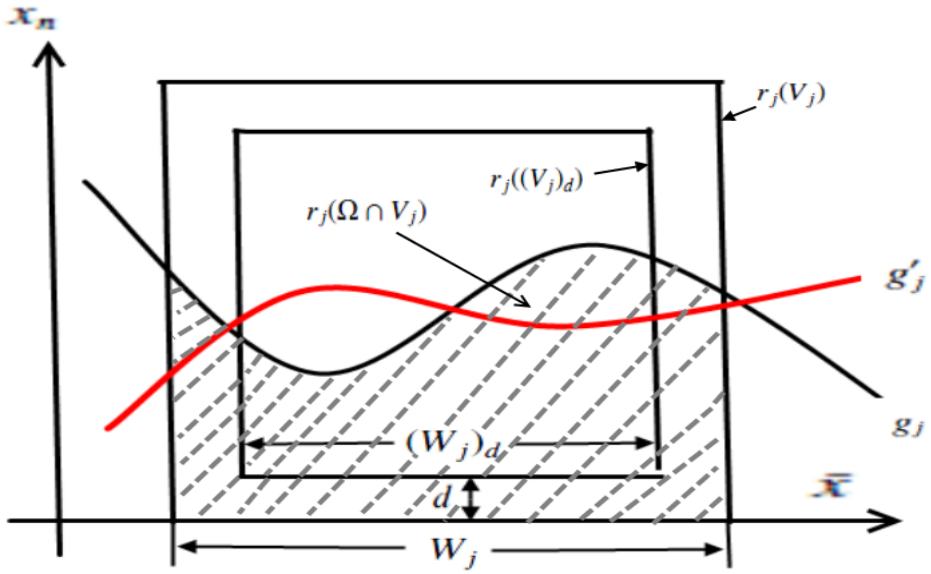
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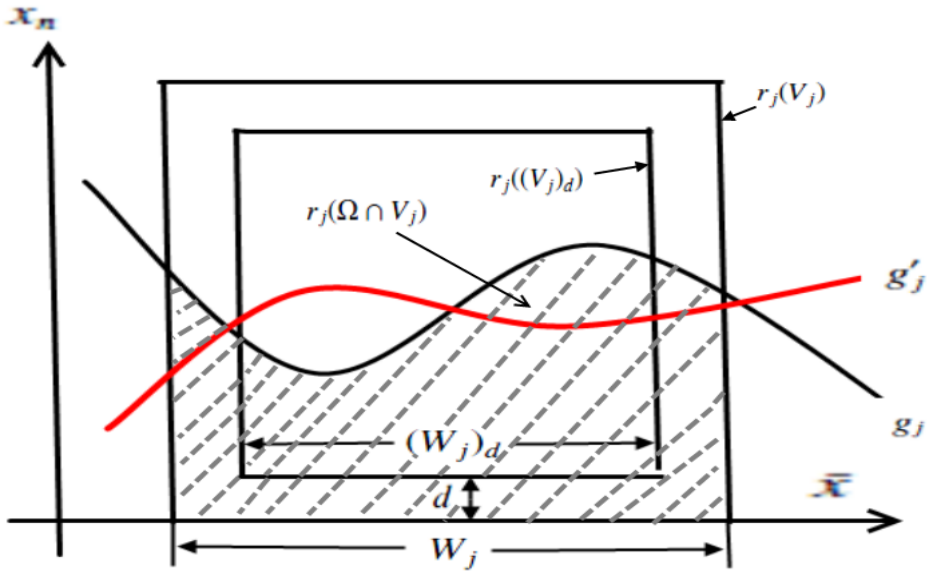
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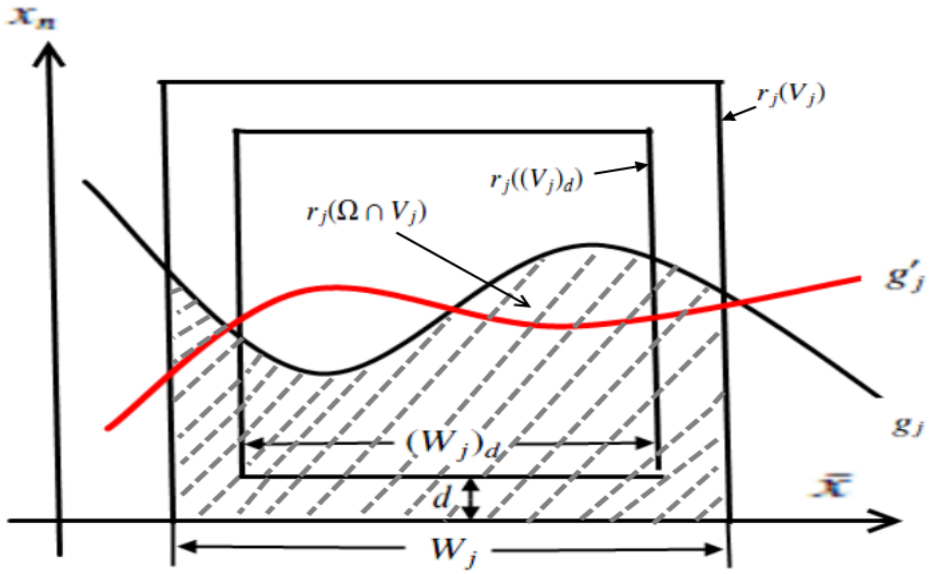
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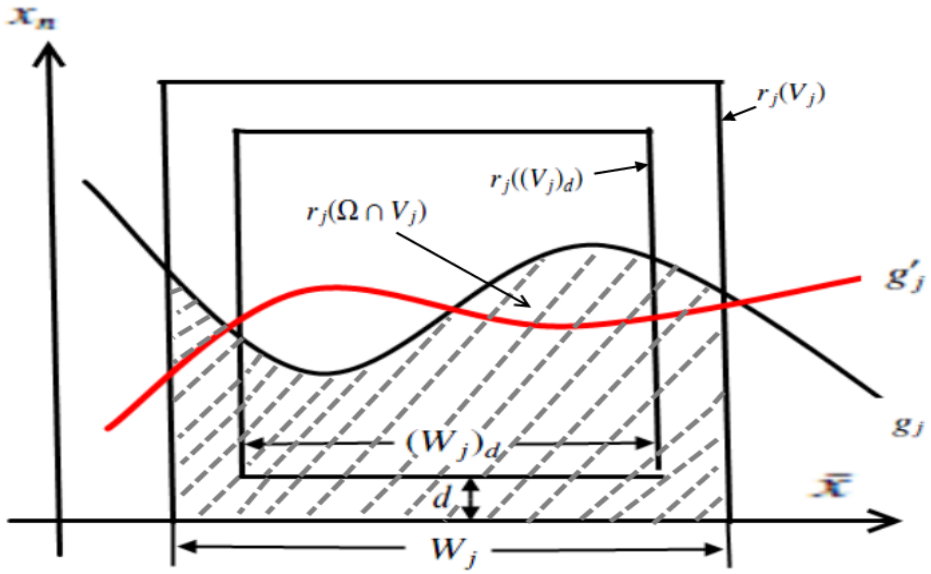
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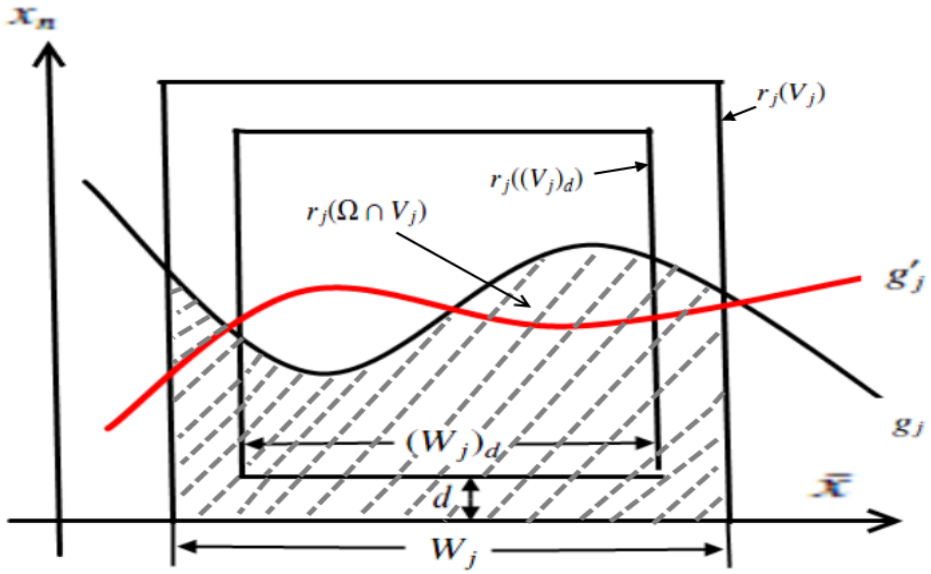
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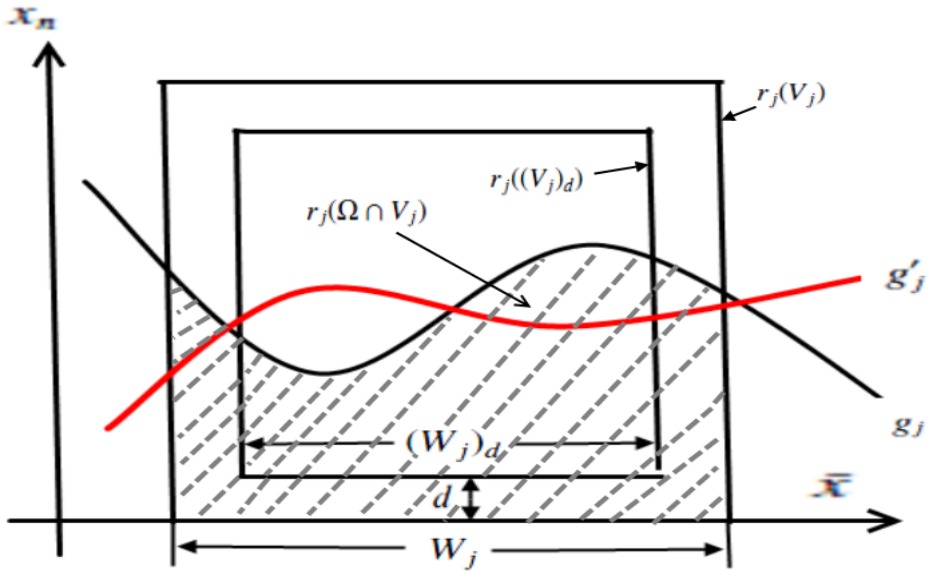
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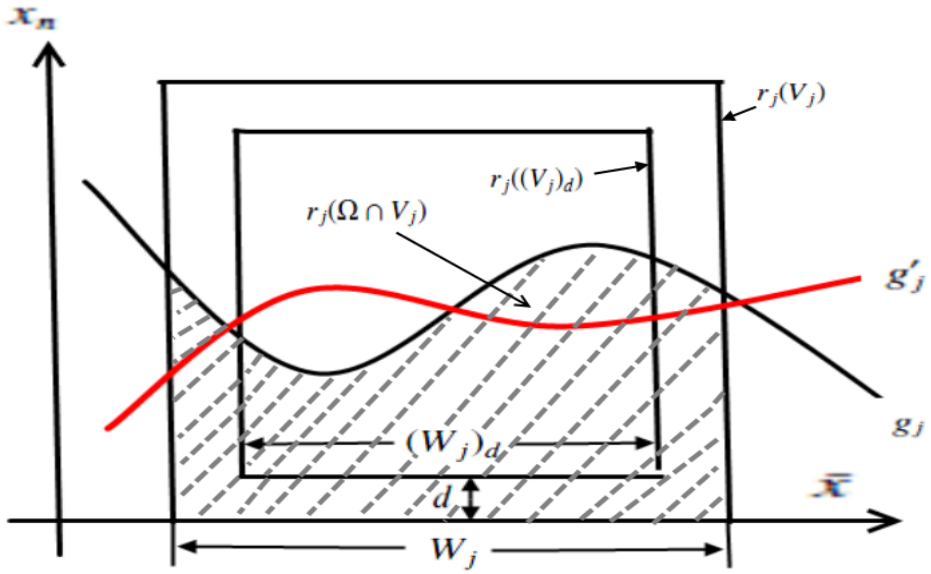
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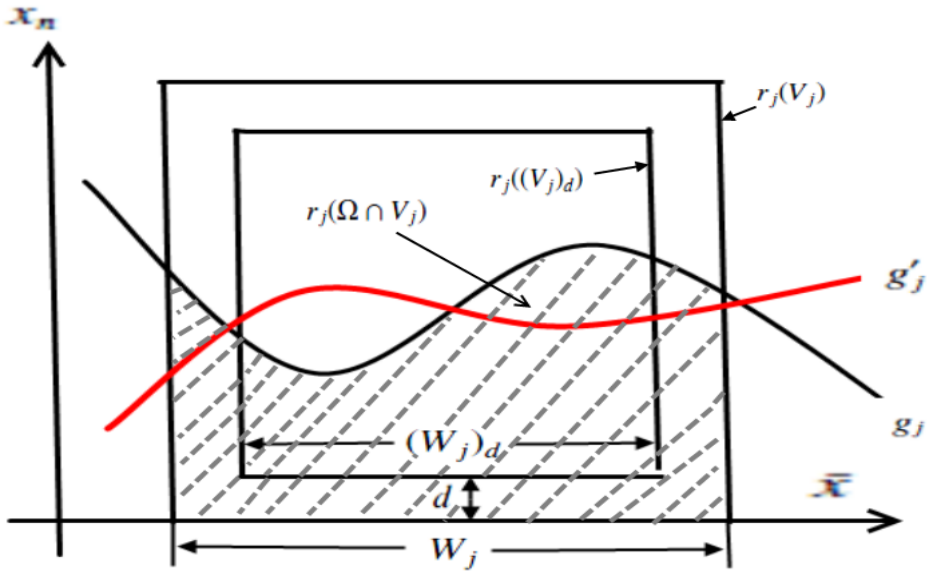
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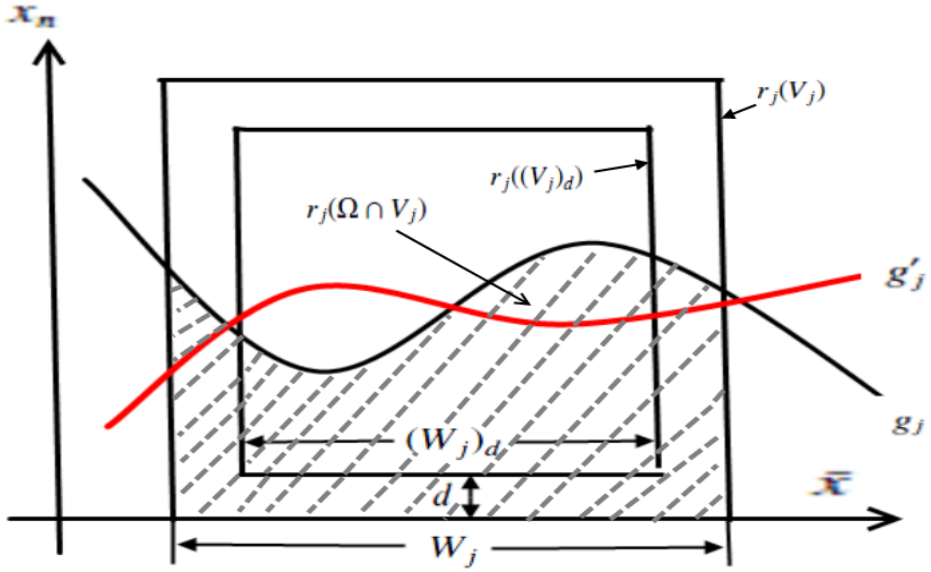
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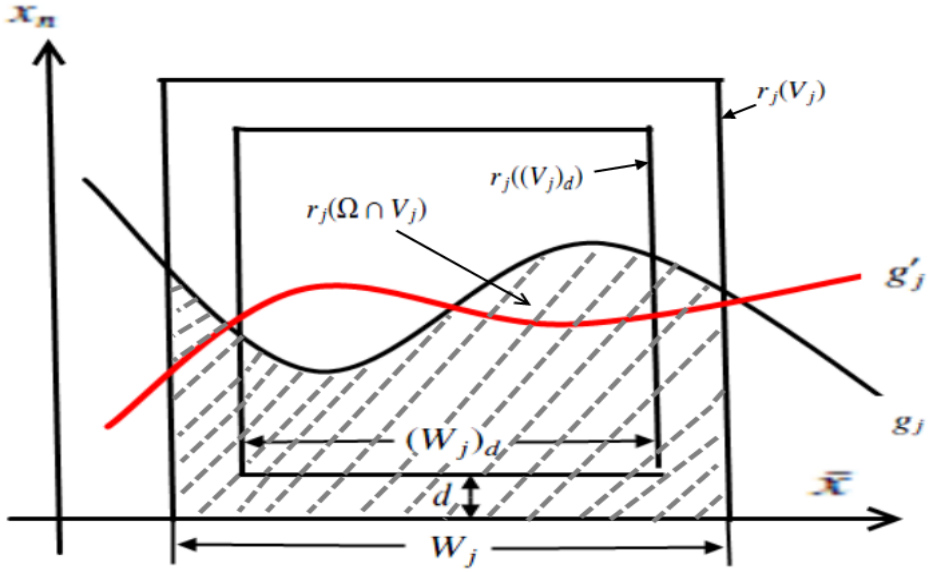
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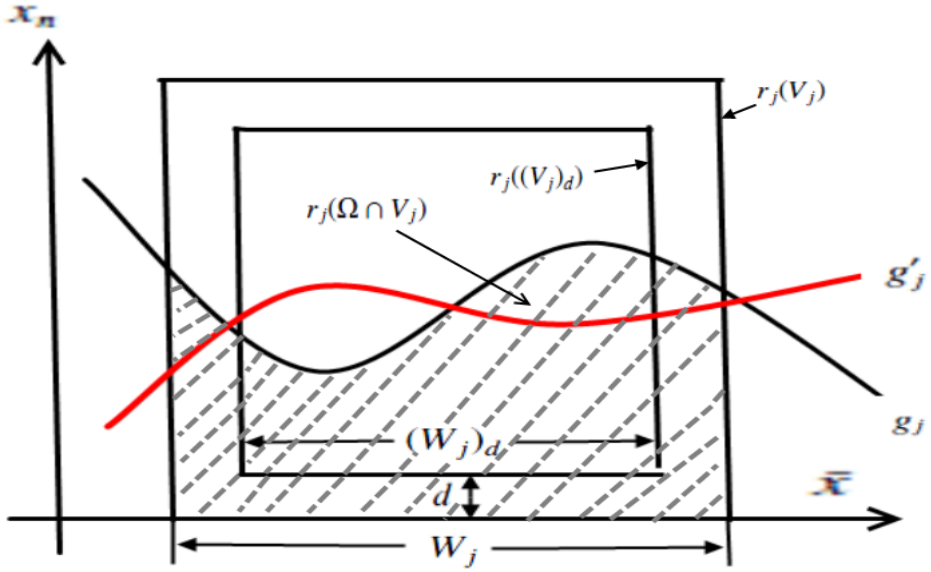
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Derivation of results 1/2

(i) Extended maximum principle of Jerrison, Kenig, Mitrea, Taylor etc..
$\exists q_1(\mathcal{F}) < 2$, s.t. for all $q_1 < q \leq \infty$, $\exists c > 0$, s.t. $\forall \Omega \in \mathcal{F}$, if $Su = 0$ in Ω, then

$$\|u\|_{L^q(\Omega)} \leq c\|u|_{\partial\Omega}\|_{L^q(\partial\Omega)}.$$

(ii) Boundary decay estimates:
For $1 \leq q \leq p \leq \infty$, $0 \leq s \leq 1$, $\exists c = c(\mathcal{A}, p, q, s) > 0$ s.t.

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(iii) Deep regularity results of Jerrison, Kenig, Mitrea, Taylor, Maz'ya etc...
For given $\mathcal{F} = C^{0,1}_M(\mathcal{A})$, $0 \leq s \leq 1$, $\exists$ an interval $(\tilde{q}_2(\mathcal{F}, s), \tilde{p}(\mathcal{F}, s))$ containing 2 s.t. if $\tilde{q}(\mathcal{F}, s) < p < \tilde{p}(\mathcal{F}, s)$, $p < 1/s$ or $p \leq 2$ if $s = 1/2$, $\exists c > 0$ s.t. $\forall \Omega \in \mathcal{F}$, $f \in W^{s-1,p}(\Omega)$, if $Su = f$ in Ω, $u = 0$ on $\partial\Omega$, then

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$$\begin{aligned}\|u - u'\|_{L^q(\Omega)} &\le c\|u'|_{\partial\Omega}\|_{L^q(\partial\Omega)} \le c \cdot d_{\mathcal{A},r}(\Omega', \Omega)^{s+1-\frac{1}{p}}\|u'\|_{W^{s+1,p}(\Omega')} \\ &\le c \cdot d_{\mathcal{A},r}(\Omega', \Omega)^{s+1-\frac{1}{p}}\|f\|_{W^{s-1,p}(D)}.\end{aligned}$$

Stability estimates for eigenfunctions: For some $c > 0$ and $\delta > 0$ that depend only on an unbounded positive selfadjoint operator T with compact resolvent and $k,\ m \in \mathbb{N}$ s.t.

$$\hat{\delta}(N_{k,m}[T], N_{k,m}[S]) \le c\|(S^{-1} - T^{-1})|_{N_{k,m}[T]}\|$$

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Results

- $\mathcal{F} = C^{0,1}_M(\mathcal{A})$ for some $M > 0$, $\mathcal{Z} = L^q(D)$, where $0 \le s \le 1$, $1 \le q \le p \le 1/s$, $2 - \varepsilon_1(M, s) < q \le p < 2 + e_2(M, s)$, for some $\varepsilon_1(M, s)$, $\varepsilon_2(M, s) > 0$ that depend only on M and s, $d(\cdot, \cdot) = d_{\mathcal{A}, r}(\cdot, \cdot)$, and r, γ are defined as above.
- $\mathcal{F} =$ domains with a uniformly Lipschitz character (e.g., $\mathcal{F} = C^{0,1}_M(\mathcal{A})$ for some $M > 0$), $\mathcal{Z} = L^2(D)$, $d(\cdot, \cdot) =$ Hausdorff distance, $\gamma = 1$.
- Ω is a domain of class C^1 and its perturbations Ω' belong to $\mathcal{G} =$ all subdomains of Ω, $\mathcal{Z} = L^\infty(D)$, $d(\cdot, \cdot) =$ Hausdorff distance, $\gamma = 1$.
- In particular: one may estimate via $d_{\mathcal{H}}(\Omega, \Omega')^{s+1-\frac{1}{q}} \cdot |\Omega \,\Delta\, \Omega'|^{\frac{1}{q}-\frac{1}{p}}$.
- If dim. $n \ge 3$, at least for $S = -\Delta$, we may take $\varepsilon_2(M, 0) > 1$; if $n = 2$, $\varepsilon_2(M, 0) > 2$.
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Results

- $\mathcal{F} = C^{0,1}_M(\mathcal{A})$ for some $M > 0$, $\mathcal{Z} = L^q(D)$, where $0 \le s \le 1$, $1 \le q \le p \le 1/s$, $2 - \varepsilon_1(M, s) < q \le p < 2 + e_2(M, s)$, for some $\varepsilon_1(M, s)$, $\varepsilon_2(M, s) > 0$ that depend only on M and s, $d(\cdot, \cdot) = d_{\mathcal{A}, r}(\cdot, \cdot)$, and r, γ are defined as above.
- $\mathcal{F} =$ domains with a uniformly Lipschitz character (e.g., $\mathcal{F} = C^{0,1}_M(\mathcal{A})$ for some $M > 0$), $\mathcal{Z} = L^2(D)$, $d(\cdot, \cdot) =$ Hausdorff distance, $\gamma = 1$.
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