Generalized semiconcavity of the value function in (stochastic) optimal control

SEMINARI DI EQUAZIONI DIFFERENZIALI E APPLICAZIONI

Padova, February 7, 2014

Ermal Feleqi

Department of Mathematics.

.



UNIVERSITÀ DEGLI STUDI DI PADOVA

ω-semiconcavity

Consider a function

$$u : K \subset X \to \mathbb{R},$$

where K is a convex subset of a normed space X.
Given a **(generalized) semiconcavity modulus**, that is an upper semicontinuous nondecreasing function s.t.

$$\omega : [0, \infty[\to [0, \infty[, \quad \omega(0+) = \lim_{\rho \to 0+} \omega(\rho) = 0,$$

we say that u is **ω-semiconcave** iff

$$\lambda u(x_1)+(1-\lambda)u(x_2)-u(\lambda x_1+(1-\lambda)x_2) \leq \lambda(1-\lambda)\|x_1-x_2\|_X \omega(\|x_1-x_2\|_X)$$

for all $x_1, x_2 \in K$, $0 \leq \lambda \leq 1$.

$C^{1,\omega}$-regularity

Properties:

- u is ω-semiconcave $+$ loc. bound. $\Rightarrow$ u is loc Lip.
- u is of class C^1 $\Leftrightarrow$ u, $-u$ are loc. ω-semiconcave for some ω, at least if $\dim(X) < \infty$.
- $\forall \alpha \in A$, u_α is ω-semiconcave $\Rightarrow$ $\inf_\alpha u_\alpha$ is ω-semiconcave.

For this and much more,
see P. Cannarsa and C. Sinestrari's book, 2004.
A map

$$f : K \subset X \to Y,$$

where X, Y are normed spaces, K is a convex subset of X, is said to be of **class** $C^{1,\omega}$ or $C^{1,\omega}$**-regular** iff

$$\|\lambda f(x_1)+(1-\lambda)f(x_2)-f(\lambda x_1+(1-\lambda)x_2)\|_Y \leq \lambda(1-\lambda)\|x_1-x_2\|_X \omega(\|x_1-x_2\|_X)$$

for all $x_1, x_2 \in X$, $0 \leq \lambda \leq 1$.

An integro-differential HJB equation

Objective: study solution

$$u : [0, T] \times \mathbb{R}^d \to \mathbb{R}$$

of

$$\begin{cases} \dfrac{\partial u}{\partial t} + \inf\limits_{\alpha \in A} \Big\{ b(t,x,\alpha) \cdot \nabla u + \dfrac{1}{2} tr\Big(\sigma(t,x,\alpha)\sigma^t(t,x,\alpha) D^2 u\Big) \\ \qquad + \displaystyle\int_{\|z\| \le \delta} \Big(u(\cdot, \cdot + H(t,z,\alpha)) - u - \nabla u \cdot H(t,z,\alpha) \Big) \nu(dz) \\ \qquad + \displaystyle\int_{\|z\| > \delta} \Big(u(\cdot, \cdot + K(t,z,\alpha)) - u \Big) \nu(dz) + L(t,x,\alpha) \Big\} = 0 \quad \text{in } [0,T) \times \mathbb{R} \\ u(T, \cdot) = \psi \quad \text{in } \mathbb{R}^d. \end{cases} \tag{HJB}$$

Arises from Bellman's approach to optimal control.
Appears in other areas of Appl. Math. such as in Mean Field Games Theory coupled with a continuity PIDE of Fokker-Planck type.

The data

The data are:

- $T > 0$ (time horizon),
- A, a metric space (the set of controls),
- ν, a Lévy meas. in $\mathbb{R}^d$, that is, Borel meas. s.t. $\int_{\mathbb{R}^d} \min\{1, \|z\|^2\}\nu(dz) < \infty$,
- $\delta > 0$ some fixed parameter,
- and maps

$$b : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^d \qquad \text{(drift)}$$

$$\sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times m} \qquad \text{(diffusion)}$$

$$H : [0,T] \times \mathbb{R}^d \times B_\delta \times A \to \mathbb{R}^d \qquad \text{(small jumps)}$$

$$K : [0,T] \times \mathbb{R}^d \times (\mathbb{R}^d \setminus B_\delta) \times A \to \mathbb{R}^d \qquad \text{(big jumps)}$$

$$L : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R} \qquad \text{(running cost or Lagrangian)}$$

$$\psi : \mathbb{R}^d \to \mathbb{R} \qquad \text{(terminal cost).}$$

A formula by Bellman

"Luckily enough" we have an "explicit" solution. But, before giving it, we must introduce a **reference probability system**

$$\Big(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\le s\le T}, \mathbb{P}, W = W(\cdot), N = N(dtdz)\Big),$$

where

- $\Big(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{0\le s\le T}, \mathbb{P}\Big)$ is a filtered probability space,
- $W = W(\cdot)$ is an m-dim. $\mathbb{F}$-Brownian motion,
- $N = N(dtdz)$ is an $\mathbb{F}$-Poisson random measure with intensity meas. ν, and compensator $\tilde{N}(dtdz) = N(dtdz) - dt\nu(dz)$,
- W and N are independent of each other.

Denote, for any $0 \le s \le T$, by $\mathcal{A}(s, T)$ the set of $\mathbb{F}$-predictable A-valued processes

$$\alpha = \alpha(\cdot) : [s, T] \times \Omega \to A \qquad \text{(admissible controls)}.$$

A formula by Bellman

Then

$$u(s,x^0) = \inf_{\alpha(\cdot)\in\mathcal{A}(s,T)} J\Big(s,x^0,\alpha(\cdot)\Big)$$

$$\text{where} \quad J\Big(s,x^0,\alpha(\cdot)\Big) = E\left[\int_s^T L\Big(t,x(t),\alpha(t)\Big)dt + \psi\Big(x(T)\Big)\right],$$

$$\begin{aligned} x(t) = x^0 &+ \int_s^t b\Big(r,x(r-),\alpha(r)\Big)dr + \int_s^t \sigma\Big(r,x(r-),\alpha(r)\Big)dW(r) \\ &+ \int_s^t \int_{\|z\|\le\delta} H\Big(r,x(r-),z,\alpha(r)\Big)\tilde{N}(drdz) \\ &+ \int_s^t \int_{\|z\|>\delta} K\Big(r,x(r-),z,\alpha(r)\Big)N(drdz). \end{aligned} \tag{IS}$$

See B. Øksendal and A. Sulem's book, 2007.

p-Hypotheses

$$\begin{cases} \|b(r,x_1,\alpha)-b(r,x_2,\alpha)\| \le C\|x_1-x_2\| \\ \|\sigma(r,x_1,\alpha)-\sigma(r,x_2,\alpha)\| \le C\|x_1-x_2\|, \\ \displaystyle\int_{\|z\|<\delta} \|H(r,x_1,z,\alpha)-H(r,x_2,z,\alpha)\|^p \nu(dz) \le C^p \|x_1-x_2\|^p, \\ \displaystyle\int_{\|z\|\ge\delta} \|K(r,x_1,z,\alpha)-K(r,x_2,z,\alpha)\|^p \nu(dz) \le C^p \|x_1-x_2\|^p, \end{cases} \tag{p-Lip}$$

and, for given semiconcavity moduli $\omega_b, \omega_\sigma, \omega_H, \omega_K$,

$$\begin{cases} \|\lambda b(r,x_1,\alpha)+(1-\lambda)b(r,x_2,\alpha)-b\big(r,x_\lambda,\alpha\big)\| \le \lambda(1-\lambda)\|x_1-x_2\|\omega_b(\|x_1-x_2\|) \\ \|\lambda \sigma(r,x_1,\alpha)+(1-\lambda)\sigma(r,x_2,\alpha)-\sigma\big(r,x_\lambda,\alpha\big)\| \le \lambda(1-\lambda)\|x_1-x_2\|\omega_\sigma(\|x_1-x_2\|) \\ \displaystyle\int_{\|z\|<\delta} \|\lambda H(r,x_1,z,\alpha)+(1-\lambda)H(r,x_1,z,\alpha)-H\big(r,x_\lambda,z,\alpha\big)\|^p \nu(dz) \\ \qquad\qquad \le (\lambda(1-\lambda)\|x_1-x_2\|\omega_H(\|x_1-x_2\|))^p \\ \displaystyle\int_{\|z\|\ge\delta} \|\lambda K(r,x_1,z,\alpha)+(1-\lambda)K(r,x_1,z,\alpha)-K\big(r,x_\lambda,z,\alpha\big)\|^p \nu(dz) \\ \qquad\qquad \le (\lambda(1-\lambda)\|x_1-x_2\|\omega_K(\|x_1-x_2\|))^p \end{cases}$$

$\forall\, 0\le r\le T,\ x_1,x_2\in\mathbb{R}^d,\ x_\lambda=\lambda x_1+(1-\lambda)x_2,\ \alpha\in A,\ 0\le\lambda\le 1$; if $p>2$, estimates for H hold also for $p=2$. (*p*-SC)

bgWave

Hypotheses on L and ψ

($2 \le p_1 \le p_2 \Rightarrow$ "p_2-hypotheses $\Rightarrow$ p_1-hypotheses".
$H = K = 0, \Rightarrow$ p-hypotheses are independent of p.)

$$\begin{cases} |L(r, x_1, \alpha) - L(r, x_2, \alpha)| \le C\,\|x_1 - x_2\|, \\ |\psi(x_1) - \psi(x_2)| \le C\,\|x_1 - x_2\|, \end{cases} \qquad \text{(Lip. costs)}$$

and, for given semiconcavity moduli ω_L, ω_ψ,

$$\begin{cases} \lambda L(r, x_1, \alpha) + (1-\lambda) L(r, x_2, \alpha) - L\big(r, \lambda x_1 + (1-\lambda) x_2, \alpha\big) \\ \qquad \le \lambda(1-\lambda)\|x_1 - x_2\| \omega_L(\|x_1 - x_2\|), \\ \lambda\psi(x_1) + (1-\lambda)\psi(x_2) - \psi\big(\lambda x_1 + (1-\lambda) x_2\big) \\ \qquad \le \lambda(1-\lambda)\|x_1 - x_2\| \omega_\psi(\|x_1 - x_2\|) \end{cases} \qquad \text{(SC costs)}$$

for all $x_1, x_2 \in \mathbb{R}^d$, $\alpha \in A$, $0 \le r \le T$, $0 \le \lambda \le 1$.

Power type moduli

Theorem (Power type moduli)

Let

$$\omega_b(\rho) = c_1\rho^{\alpha_1}, \quad \omega_\sigma(\rho) = c_2\rho^{\alpha_2}, \quad \omega_H(\rho) = c_3\rho^{\alpha_3},$$
$$\omega_K(\rho) = c_4\rho^{\alpha_4}, \quad \omega_L(\rho) = c_5\rho^{\alpha_5}, \quad \omega_\psi(\rho) = c_6\rho^{\alpha_6} \qquad \forall \rho \geq 0,$$

where $c_i \geq 0$, $0 < \alpha_i \leq 1$, $i = 1, \dots, 6$. *Assume that p-hypotheses hold for* $p = 2 + 2\max\{\alpha_i \ : \ i = 1, \dots, 4\}$, *and that hypotheses on* L *and* ψ *are also satisfied.*
Then, for all $0 \leq s \leq T$, $u(s, \cdot)$ *is* ω*-semiconcave, where*

$$\omega(\rho) = \sum_{i=1}^{6} c_i'\rho^{\alpha_i}, \qquad \forall \rho \geq 0,$$

where constants c_i' depend only on $T, \nu(\mathbb{R}^d \setminus B_\delta)$, Lip. constants of b, σ, H, K, L, ψ, c_i, α_i, $i = 1, \cdots, 6$

Theorem (By continuity on initial conditions via Kolmogorov's criterion)

Assume that p-hypotheses hold for some $p > d$, and let $0 < \beta < 1 - d/p$. Assume that hypotheses on L and ψ are also satisfied. Then, for all $0 \le s \le T$, $u(s,\cdot)$ is ω-semiconcave, where

$$\begin{aligned}\omega(\rho) = c_1'\omega_b(c_1\omega_c(\rho)) + c_2'\omega_\sigma(c_2\omega_c(\rho)) + c_3'\omega_H(c_3\omega_c(\rho))\\ +c_4'\omega_K(c_4\omega_c(\rho)) + c_5'\omega_L(c_5\omega_c(\rho)) + c_6'\omega_\psi(c_6\omega_c(\rho)),\end{aligned}$$

where constants c_i, c_i', $i = 1,\dots,6$, depend only on p, T, $\nu(\mathbb{R}^d \setminus B_\delta)$, Lip. constants of b, σ, H, K, L, ψ, with

$$\omega_c(\rho) = c\,(\rho^\beta + \rho), \quad \forall \rho \ge 0,$$

where c depends only on β and d.

Burkholder-Davis-Gundy inequalities

Let $2 \leq p < \infty$. Then

$$E\left[\left\|\int_s^t \sigma(r)dW(r)\right\|^p\right] \leq c'_p E\left[\left(\int_s^t \|\sigma(r)\|^2 dr\right)^{p/2}\right],$$

$$E\left[\left\|\int_s^t \int_{\|z\|<\delta} H(r,z)\tilde{N}(drdz)\right\|^p\right] \leq c_p E\left[\int_s^t \int_{\|z\|<\delta} \|H(r,z)\|^p dr\nu(dz)\right] + c''_p E\left[\left(\int_s^t \int_{\|z\|<\delta} \|H(r,z)\|^2 dr\nu(dz)\right)^{p/2}\right].$$

Actually, for $p = 2$, they are "=" with $c_p = c'_p = 1$ and $c''_p = 0$.
Consequences:

$$E\left[\left\|\int_s^t \int_{E^c} K(r,z)N(drdz)\right\|^p\right] \leq cE\left[\int_s^t \int_{\|z\|>\delta} \|K(r,z)\|^p dr\nu(dz)\right]$$

To a deterministic problem via BDG ineq

Let $B = L^p(\Omega, \mathcal{F}, \mathbb{P}; \mathbb{R}^d)$ and, for any càdlàg $x(\cdot) \in L^p([s,T] \times \Omega\ \mathbb{R}^d)$, let $\Phi_s\Big(x(\cdot)(t)\Big)$ is RHS of (IS) $-x^0$. Then

$$\begin{aligned}\|\Phi_s\Big(x(\cdot)\Big)(t)\|_B^p \leq C \int_s^t \Big(& \|b(r, x(r-), \alpha(r))\|_B^p \\ & + \|\sigma(r, x(r-), \alpha(r))\|_B^p + \|H(r, x(r-), \cdot, \alpha(r))\|^p_{L^p\left(\Omega,\mathcal{F},\mathbb{P};L^2(B_\delta,\nu;\mathbb{R}^d)\right)} \\ & + \|H(r, x(r-), \cdot, \alpha(r))\|^p_{L^p\left(\Omega,\mathcal{F},\mathbb{P};L^p(B_\delta,\nu;\mathbb{R}^d)\right)} \\ & + \|K(r, x(r-), \cdot, \alpha(r))\|^p_{L^p\left(\Omega,\mathcal{F},\mathbb{P};L^p(\mathbb{R}^d\setminus B_\delta,\nu;\mathbb{R}^d)\right)} \Big) dr.\end{aligned}$$

Let

$$\overline{L}(t,x,\alpha) = E[L(t,x,\alpha)], \qquad \overline{\psi}(x) = E[\psi(x)] \qquad \text{(new costs)}$$

for all $x \in B$, $\alpha \in \mathcal{A}$–the set of A-valued random variables.

An “abstract” optimal control problem

It makes sense to consider

$$V(s,x^0) = \inf_{\alpha(\cdot)\in\mathcal{A}(s,T)} J\Big(s,x^0,\alpha(\cdot)\Big), \qquad \text{(val. func.)}$$

where

$$J\Big(s,x^0,\alpha(\cdot)\Big) = \int_s^T \overline{L}\Big(t,x(t),\alpha(t)\Big)dt + \overline{\psi}\Big(x(T)\Big), \qquad \text{(cost)}$$

$T > 0$, $\mathcal{A}$-a metric space, $\mathcal{A}(s,T)$-a set of maps $\alpha(\cdot) : [s,T] \to \mathcal{A}$, $C \hookrightarrow B$ are normed spaces

$$\overline{L} : [s,T]\times C\times\mathcal{A} \to \mathbb{R}, \qquad \overline{\psi} : C \to \mathbb{R} \qquad \text{(RTC)}$$

given functional,

$$x(\cdot) = x^0 + \Phi_s\Big(x(\cdot),\alpha(\cdot)\Big), \quad \Phi_s : \Sigma(s,T) \to \Sigma(s,T), \qquad \text{(dyn)}$$

$x^0 \in C \subset B$, $\Sigma(s,T)$ a linear space of maps $x(\cdot) : [s,T] \to C \subset B$, called admissible trajectories, which is (p, SC)-**compatible** with respect to maps

$$\bar{f} : [0,T]\times B\times\mathcal{A} \to B', \quad \bar{f} : [0,T]\times C\times\mathcal{A} \to C'$$

where $C \hookrightarrow B'$ are other normed spaces.

A (p, SC)-**compatible** opt. ctrl. problem

$$\left\|\lambda\Phi_s\big(x_1(\cdot),\alpha(\cdot)\big)(t)+\mu\Phi_s\big(x_2(\cdot),\alpha(\cdot)\big)(t)-\Phi_s\big(x_3(\cdot),\alpha(\cdot)\big)(t)\right\|_B^p$$

$$\leq \int_s^t \left\|\lambda\bar{f}\big(r,x_1(r),\alpha(r)\big)+\mu\bar{f}\big(r,x_2(r),\alpha(r)\big)-\bar{f}\big(r,x_3(r),\alpha(r)\big)\right\|_{B'}^p dr.$$

for all $0 \leq s \leq t \leq T$, $x_1(\cdot), x_2(\cdot), x_3(\cdot) \in \Sigma(s,T)$, $\alpha(\cdot) \in \mathcal{A}(s,T)$, and $0 \leq \lambda, \mu \leq 1$.

Lemma ((p, SC)-compatibility by BDG ineq.)

The optimal control problem introduced previously is (p, SC)-compatible w.r.t. $c\,\bar{b} \times \bar{\sigma} \times \bar{H}_1 \times \bar{H}_2 \times \bar{K}$, where $c > 0$,

$$\bar{b} : [0,T] \times L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \times \mathcal{A} \to L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d),$$

$$\bar{\sigma} : [0,T] \times L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \times \mathcal{A} \to L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^{d\times m}),$$

$$\bar{H}_1 : [0,T] \times [0,T] \times L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \times \mathcal{A} \to L^p(\Omega \times B_\delta, \mathcal{F} \otimes \mathcal{B}(B_\delta), \mathbb{P} \otimes \nu($$

$$\bar{H}_2 : [0,T] \times L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \times \mathcal{A} \to L^p\Big(\Omega,\mathcal{F},\mathbb{P};L^2(B_\delta,\mathcal{B}(B_\delta),\nu(dz);\mathbb{R}^d)\Big)$$

$$\bar{K} : [0,T] \times L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d) \times \mathcal{A} \to L^p(\Omega \times (\mathbb{R}^d \setminus B_\delta), \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^d \setminus B_\delta), \mathbb{P} \otimes$$

A (p, SC)-**compatible** opt. ctrl. problem

$$\left\|\lambda\Phi_s\big(x_1(\cdot),\alpha(\cdot)\big)(t)+\mu\Phi_s\big(x_2(\cdot),\alpha(\cdot)\big)(t)-\Phi_s\big(x_3(\cdot),\alpha(\cdot)\big)(t)\right\|_B^p$$
$$\leq\int_s^t\left\|\lambda\bar{f}\big(r,x_1(r),\alpha(r)\big)+\mu\bar{f}\big(r,x_2(r),\alpha(r)\big)-\bar{f}\big(r,x_3(r),\alpha(r)\big)\right\|_{B'}^p dr.$$

for all $0\leq s\leq t\leq T$, $x_1(\cdot)$, $x_2(\cdot)$, $x_3(\cdot)\in\Sigma(s,T)$, $\alpha(\cdot)\in\mathcal{A}(s,T)$, and $0\leq\lambda,\mu\leq 1$.

Lemma ((p, SC)-compatibility by BDG ineq.)

The optimal control problem introduced previously is (p, SC)-compatible w.r.t. $c\,\bar{b}\times\bar{\sigma}\times\bar{H}_1\times\bar{H}_2\times\bar{K}$, where $c>0$,

$$\bar{b}(t,x,\alpha)(\omega)=b(t,x(\omega),\alpha(\omega)),\quad\bar{\sigma}(t,x)(\omega,\alpha)=\sigma(t,x(\omega),\alpha(\omega)),$$
$$\bar{H}_1(t,x,\alpha)(\omega,z)=H(t,x(\omega),z,\alpha(\omega)),\quad H_2(t,x,\alpha)(\omega)=H(t,x,\cdot,\alpha(\omega)),$$
$$\bar{K}(t,x)(\omega,z^c)=K(t,x(\omega),z^c),$$

for all $0\leq t\leq T$, $x\in L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^d)$, $\alpha\in\mathcal{A}$, $\omega\in\Omega$. $z\in B_\delta$, $z^c\in\mathbb{R}^d\setminus B_\delta$.

Main result

Theorem (Semiconcavity of the value function)

Let be given normed spaces B, C, B', C' such that

$$C \hookrightarrow B, \qquad C' \hookrightarrow B'.$$

Let $\Phi = \{\Phi_s : \Sigma(x,T) \to \Sigma(s,T)\}_{0\le s\le T}$ be (p_1, SC)-compatible w.r.t.

$$\bar{f} : [0,T] \times B \times \mathcal{A} \to B'$$

for some $1 \le p_1 < \infty$, and (p_2, SC)-compatible w.r.t.

$$\bar{f} : [0,T] \times C \times \mathcal{A} \to C'$$

for some $1 \le p_2 < \infty$. Assume that for some $C > 0$ and some semiconcavity modulus ω_f

$$\left.\begin{aligned} \|\bar{f}(t,x_1,\alpha) - \bar{f}(t,x_2,\alpha)\|_{C'} &\le C\|x_1 - x_2\|_C \\ \|\lambda\bar{f}(t,x_1,\alpha) + (1-\lambda)\bar{f}(t,x_2,\alpha) - \bar{f}(t,x_\lambda,\alpha)\|_{B'} &\\ \le \lambda(1-\lambda)\|x_1 - x_2\|_C\omega_{\bar{f}}(\|x_1 - x_2\|_C &\end{aligned}\right\}$$

$$\begin{aligned} &(\bar{f}(t,\cdot,\alpha) \text{ is Lip.}), \\ &\left(\bar{f}(t,\cdot,\alpha) \in C^{1,\omega_{\bar{f}}}\right), \end{aligned}$$

for all $x_1, x_2 \in C$, $0 \le \lambda \le 1$, $\alpha \in \mathcal{A}$, $0 \le t \le T$, $x_\lambda = \lambda x_1 + (1-\lambda)x_2$.

Main result

Theorem (Semiconcavity of the value function)

Assume also that $\overline{L}(t,\cdot,\alpha)$ is $\omega_{\overline{L}}$-semiconcave and $\overline{\psi}$ is $\omega_{\overline{\psi}}$-semiconcave, and both are globally Lipschitz, in C, where

$$\overline{L} : [0,T] \times C \to \mathcal{A} \to \mathbb{R}, \qquad \overline{\psi} : C \to \mathbb{R}.$$

Then $V(s,\cdot)$ is Lipschitz and ω- semioncave in C with

$$\omega(\rho) = c_1' \omega_{\overline{f}}(c_1\rho) + c_2' \omega_{\overline{L}}(c_2\rho) + c_3' \omega_{\overline{\psi}}(c_3\rho),$$

with c_i, c_i' that depend only on p_1, p_2, T, Lip. const.s (and sem.conc. mod.) of $\overline{f}$, $\overline{L}$, $\overline{\psi}$

Proof. Let $x_i(\cdot)$, $i = 1,2,3$, be solutions to $\Phi_s(\cdot,\alpha(\cdot))$ corresponding to initial conditions $x^0 = x_i$, $i = 1,2$, $x^0 = \lambda x_1^0 + (1-\lambda)x_2^0$, and to a fixed control $\alpha(\cdot) \in \mathcal{A}(s,T)$.

Proof.

$$\lambda J\big(s,x_1^0,\alpha(\cdot)\big)+(1-\lambda)J\big(s,x_2^0,\alpha(\cdot)\big)-J\big(s,x_3^0,\alpha(\cdot)\big)$$

$$=\int_s^T\Big(\lambda\overline{L}\big(t,x_1(t),\alpha(t)\big)+(1-\lambda)\overline{L}\big(t,x_2(t),\alpha(t)\big)-\overline{L}\big(t,\lambda x_1(t)+(1-\lambda)x_2(t),\alpha(t)\big)\Big)dt$$

$$+\lambda\overline{\psi}\big(x_1(T)\big)+(1-\lambda)\overline{\psi}\big(x_2(T)\big)-\overline{\psi}\big(\lambda x_1(T)+(1-\lambda)x_2(T)\big)$$

$$+\int_s^T\Big(\overline{L}\big(t,\lambda x_1(t)+(1-\lambda)x_2(t),\alpha(t)\big)-L\big(t,x_3(t),\alpha(t)\big)\Big)dt$$

$$+\overline{\psi}\big(\lambda x_1(T)+(1-\lambda)x_2(T)\big)-\overline{\psi}\big(x_3(T)\big)$$

$$\leq\lambda(1-\lambda)\Big(\int_s^T\|x_1(t)-x_2(t)\|_C\omega_{\overline{L}}(\|x_1(t)-x_2(t)\|_C)dt$$

$$+\|x_1(T)-x_2(T)\|_C\omega_{\overline{\psi}}(\|x_1(T)-x_2(T)\|_C)\Big)+C\Big(\int_s^T\|\lambda x_1(t)+(1-\lambda)x_2(t)-x_3(t)\|_B dt$$

$$\|\lambda x_1(T)+(1-\lambda)x_2(T)-x_3(T)\|_B\Big).$$

We need $\sup_{s\leq t\leq T}\|x_1(t)-x_2(t)\|_C\leq C\,\|x_1^0-x_2^0\|_C$, $\sup_{s\leq t\leq T}\|\lambda x_1(t)+(1-\lambda)x_2(t)-x_3(t)\|_B$

$\leq\lambda(1-\lambda)C\,\|x_1^0-x_2^0\|_C\omega_f(\|x_1^0-x_2^0\|_C)$ to conclude.

Applications

Let

- $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete prob. space,
- X a normed space–the state space ($X = \mathbb{R}^d$),
- B a normed space of X-valued random variables ($B = L^2(\Omega; X)$),
- Y another normed space
  $(Y = \mathbb{R}^d \times \mathbb{R}^{d\times m} L^2(B_\delta, \nu; \mathbb{R}^d) \times L^p(B_\delta, \nu; \mathbb{R}^d) \times L^p(\mathbb{R}^d \setminus B_\delta, \nu; \mathbb{R}^d))$,
- B' a normed space of Y-valued random variables ($L^p(\emptyset; Y)$, $p \geq 2$),
- A a metric space–set of controls,
- $\mathcal{A}$ set of A-valued random variables.

Assume

$$\bar{f}(t, x, \alpha)(\omega) = f(t, x(\omega), \alpha(\omega)), \quad \overline{L}(x) = E[L(t, x, \alpha)], \quad \overline{\psi}(x) = E[\psi(x)]$$

for all $0 \leq t \leq T$, $x \in B$, ($x \in C$ in case of $\overline{L}$, $\overline{\psi}$) $\alpha \in \mathcal{A}$, $\omega \in \Omega$, where

$$f : [0, T] \times X \times A \to Y, \quad L : [0, T] \times X \times A \to \mathbb{R}, \quad \psi : X \to \mathbb{R}.$$

Assume f is C^{1,ω_f}-regular, L is ω_L-semiconcave, ψ is ω_ψ-semiconcave, uniformly in time and control variables.

Lemma (invariance of power type moduli)

$\omega_f(\rho) = c\,\rho^\alpha$ *with* $\Rightarrow \omega_{\bar{f}} = \omega_f$ *if* $L^2(\Omega; Y) \hookrightarrow B'$, *and* $C \hookrightarrow L^{2(1+\alpha)}(\Omega; X)$.
$\omega_L(\rho) = c\,\rho^\alpha \Rightarrow \omega_{\bar{L}} = \omega_L$ *if* $C \hookrightarrow L^{1+\alpha}(\Omega; X)$.

Lemma (moduli with certain concavity properties)

Let

$$\gamma_f(\rho) = \left(\rho^{\beta_1}\omega_f^2(\rho)\right)^{q_1}, \quad \gamma_L(\rho) = \left(\rho^{\beta_2}\omega_L(\rho)\right)^{q_2}, \quad \gamma_\psi = \left(\rho^{\beta_3}\omega_\psi(\rho)\right)^{q_3}, \quad \forall \rho \geq 0,$$

where $0 \leq \beta_i \leq 1$, $1 \leq q_i \leq \infty$, $i = 1, 2, 3$. *Let also* $r_i^{-1} + q_i^{-1} = 1$. *Then*

- γ_f *concave and* $L^2(\Omega; Y) \hookrightarrow B'$, $C \hookrightarrow L^{(2-\beta_1)r_1}(\Omega; X)$, $C \hookrightarrow L^1(\Omega; X) \Rightarrow \omega_{\bar{f}} = \omega_f$.
- γ_L *concave and* $C \hookrightarrow L^{(1-\beta_1)r_2}(\Omega; X) \Rightarrow \omega_{\bar{L}} = \omega_L$.
- γ_ψ *concave and* $C \hookrightarrow L^{(1-\beta_1)r_3}(\Omega; X) \Rightarrow \omega_{\bar{\psi}} = \omega_\psi$.

Proof. Based on Jensen's inequality for concave $\gamma : [0, \infty[\to [0, \infty[$

$$E[\gamma(\|x\|)] \leq \gamma(E[\|x\|]), \quad \forall x \in L^1(\Omega; X).$$

bgWave

An improvement

For fixed $\alpha(\cdot)$, let $x(\cdot, x^0)$ denote a sol. to Φ_s for in. cond. x^0.

Remark (An improvement)

Actually, for the main result to remain valid, we need only

- *ω-semiconcavity (and Lipsch.) of $\overline{L}, \overline{\psi}$,*
- *$C^{1,\omega}$-regularity (and Lipsch.) of $\overline{f}$*

along admissible solutions.
E.g.

$$\|\lambda\overline{f}\big(r, x(r, x_1^0), \alpha(r)\big) + (1-\lambda)\overline{f}\big(r, x(r, x_1^0), \alpha(r)\big) - \overline{f}\big(r, \lambda x(r, x_1^0) + (1-\lambda)x(r, x_2^0), \alpha(r)\big)\|_{B'} \leq \lambda(1-\lambda)\|x_1^0 - x_2^0\|_C \omega_f(\|x_1^0 - x_2^0\|_C).$$

Via Kolmogorov's continuity criterion

Fixed $(s, x^0) \in [0, T] \times C$, let

$x(\cdot, x^0) : [s, T] \to C \subset B$ be a fixed point of $\Phi_s(, \alpha(\cdot)) + x^0$.

Assume that for some continuity modulus ω_c

$$\|x(t, x_1^0) - x(t, x_2^0)\|_X \leq \omega_c(\|x_1^0 - x_2^0\|_C) \quad \forall x_1^0, X_2^0 \in C, \ \mathbb{P} - a.s. \qquad \text{(CT)}$$

Lemma

Then, along trajectories,

- $\omega_{\overline{f}} = \omega_f \circ \omega_c, \quad \omega_{\overline{L}} = \omega_L \circ \omega_c, \quad \omega_{\overline{\psi}} = \omega_\psi \circ \omega_c..$

Theorem (Kolmogorov's continuity criterion)

Suppose that C is d-dimensional and, that for some $p > d$, $c > 0$,

$$E[\|x(t, x_1^0) - x(t, x_2^0)\|_X^p] \leq C \|x_1^0 - x_2^0\|_C^p, \quad \forall s \leq t \leq T, \ x_1^0, x_2^0 \in C.$$

Then, for any $0 \leq \beta < 1 - d/p$ (CT) holds with $\omega_c(\rho) = c\,(\rho^\beta + \rho)$, $\rho \geq 0$, where c depends only on β, and d.

Thank you for being here.

THE END