Towards a useful set-valued iterated Lie bracket

Analysis and Geometry in Control Theory and its Applications (with a special tribute to Hélène Frankowska and Hector J. Sussmann)

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Outline

1 Asymptotic formulas

2 Integral formulas

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4 Set-valued iterated bracket

5 A nonsmooth Chow-Rashevski's theorem

Usefulness of (iterated) Lie brackets

- Commutativity theorem,
- Frobenius's theorem
- Chow-Rashevski's controllability theorem '38-'39,
  - Together with Bony's theorem '69, yields strong maximum principle for hypoelliptic linear PDEs;
  - (In passing, for more on max. principles for PDEs, see papers by Bardi and Da Lio, and references therein,)
- Hörmander's theorem on hypoelliptic linear PDEs '67
  - See M. Bramanti's book, "An invitation to..." (2013).

Asymptotic formulas

These are classical (at least for vector fields smooth enough).

- If $f_1 \in C^0$, and . . . ,

$$xe^{tf_1} = x + t_1 f_1(x) + o(t_1)$$

as $t_1 \to 0$.

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- If $f_1, f_2 \in C^1$,

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} = x + t_1 t_2 [f_1, f_2](x) + o(t_1 t_2) \qquad \text{(SAF)}$$

  as $(t_1, t_2) \to (0, 0)$.

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$$xe^{tf_1} = x + t_1 f_1(x) + o(t_1)$$

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as $(t_1, t_2) \to (0, 0)$.

- If $f_1, f_2, \in C^2$, $f_3 \in C^1$,

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{t_2 f_2} e^{t_1 f_1} e^{-t_2 f_2} e^{-t_1 f_1} e^{-t_3 f_3} = x + t_1 t_2 t_3 \left[[f_1, f_2], f_3\right](x) + o(t_1 t_2 t_3)$$

as $(t_1, t_2, t_3) \to (0, 0, 0)$.

Asymptotic formulas (continued)

- If $f_1, f_2, \in C^2$, $f_3, f_4, \in C^2$,

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{t_4 f_4} e^{-t_3 f_3} e^{-t_4 f_4} e^{t_2 f_2} e^{t_1 f_1} e^{-t_2 f_2} e^{-t_1 f_1}$$
$$e^{t_4 f_4} e^{t_3 f_3} e^{-t_4 f_4} e^{-t_3 f_3} = x + t_1 t_2 t_3 t_4 \left[[f_1, f_2], [f_3, f_4]\right](x) + o(t_1 t_2 t_3 t_4)$$

  as $(t_1, t_2, t_3, t_4) \to (0, 0, 0, 0)$.

See Brandolini, Bramanti, and Pedroni (2013).
However, not under as weak regularity as stated here, but with all $f_i \in C^{deg(B)-1}$.
A notion of formal bracket B can be introduced, $deg(B)$ is defined.

Asymptotic formulas (continued)

- If $f_1, f_2, \in C^2$, $f_3, f_4, \in C^2$,

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{t_4 f_4} e^{-t_3 f_3} e^{-t_4 f_4} e^{t_2 f_2} e^{t_1 f_1} e^{-t_2 f_2} e^{-t_1 f_1}$$
$$e^{t_4 f_4} e^{t_3 f_3} e^{-t_4 f_4} e^{-t_3 f_3} = x + t_1 t_2 t_3 t_4 \left[[f_1, f_2], [f_3, f_4]\right](x) + o(t_1 t_2 t_3 t_4)$$

as $(t_1, t_2, t_3, t_4) \to (0, 0, 0, 0)$.

See Brandolini, Bramanti, and Pedroni (2013).
However, not under as weak regularity as stated here, but with all $f_i \in C^{deg(B)-1}$.
A notion of formal bracket B can be introduced, $deg(B)$ is defined.
Let $B = [B_1, B_2]$ with $deg(B) = m$, $deg(B) = m_1$, $deg(B_2) = m_2$, so that $m_1 + m_2 = m$. Let $\mathbf{f} = (f_1, \ldots, f_m)$.
Suppose

$$B(\mathbf{f}) = [B_1(\mathbf{f}_{(1)}), B_2(\mathbf{f}_{(2)})],$$

where

$$\mathbf{f}_{(1)} = (f_1, \ldots, f_{m_1}), \qquad \mathbf{f}_{(2)} = (f_{m_1+1}, \ldots, f_m).$$

Asymptotic formulas (continued)

Define recursively

$$\Psi_B^{\mathbf{f}}(t_1,\dots,t_m) = \Psi_{B_1}^{\mathbf{f}^{(1)}}(t_1,\dots,t_{m_1})\Psi_{B_2}^{\mathbf{f}^{(2)}}(t_{m_1+1},\dots,t_m) \left(\Psi_{B_1}^{\mathbf{f}^{(1)}}(t_1,\dots,t_{m_1})\right)^{-1}\left(\Psi_{B_2}^{\mathbf{f}^{(2)}}(t_{m_1+1},\dots,t_m)\right)^{-1}$$

Theorem (Asymptotic formulas)

If $\mathbf{f} \in C^B$, *then*

$$x\Psi_B^{\mathbf{f}}(t_1,\dots,t_m) = x + t_1\cdots t_m\,(xB(\mathbf{f})) + o(t_1\cdots t_m)$$

as $(t_1,\dots,t_m) \to 0_{R^m}$.

Integral formulas

- $$xe^{t_1 f_1} = x + \int_0^{t_1} xe^{s_1 f_1} ds_1.$$
- Rampazzo and Sussmann (2001), (2007)

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} = x + \int_0^{t_1} \int_0^{t_2} xe^{t_1 f_1} e^{s_2 f_2} e^{(s_1 - t_1) f_1} [f_1, f_2] e^{-s_1 f_1} e^{-s_2 f_2} ds_1\, ds_2.$$

- Rampazzo and F. (2014)

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{t_2 f_2} e^{t_1 f_1} e^{-t_2 f_2} e^{-t_1 f_1} e^{-t_3 f_3} - x =$$
$$\int_0^{t_1} \int_0^{t_2} \int_0^{t_3} xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{s_3 f_3} e^{t_2 f_2} e^{t_1 f_1} e^{(s_2 - t_2) f_2} e^{-t_1 f_1} e^{-s_2 f_2}$$
$$\left[e^{s_2 f_2} e^{s_1 f_1} [f_1, f_2] e^{-s_1 f_1} e^{-s_2 f_2},\ f_3 \right] e^{s_2 f_2} e^{t_1 f_1} e^{-s_2 f_2} e^{-t_1 f_1} e^{-s_3 f_3} ds_1\, ds_2\, ds_3.$$

Integral formulas (continued)

$$xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{t_4 f_4} e^{-t_3 f_3} e^{-t_4 f_4} e^{t_2 f_2} e^{t_1 f_1} e^{-t_2 f_2} e^{-t_1 f_1} e^{t_4 f_4} e^{t_3 f_3} e^{-t_4 f_4} e^{-t_3 f_3} - x =$$

$$\int_0^{t_1}\int_0^{t_2}\int_0^{t_3}\int_0^{t_4} xe^{t_1 f_1} e^{t_2 f_2} e^{-t_1 f_1} e^{-t_2 f_2} e^{t_3 f_3} e^{s_4 f_4} e^{-t_3 f_3} e^{-s_4 f_4} e^{t_2 f_2} e^{t_1 f_1} e^{(s_2 - t_2) f_2} e^{-t_1 f_1} e^{-s_2 f_2} \Big[e^{s_2 f_2}$$

$$e^{s_1 f_1} [f_1, f_2] e^{-s_1 f_1} e^{-s_2 f_2} \,,\; e^{s_4 f_4} e^{s_3 f_3} [f_3, f_4] e^{-s_3 f_3} e^{-s_4 f_4} \Big] e^{t_2 f_2} e^{t_1 f_1} e^{-s_2 f_s} e^{-t_1 f_1} e^{-s_3 f_3} \, ds_1 \, ds_2 \, ds_3 \, ds_4.$$

Theorem (Integral formulas)

Let $\mathbf{f} = (f_1, \ldots, f_m) \in C^B$. *There are multiflows, that is, products of*

$$\{e^{t_i f_i}\}, \quad \{e^{s_j f_j}\},$$

and their inverses, as many as is the number of subbrackets B' *of* B *of* $\deg(B') \geq 2$, *say* r, *and call them* $\Phi_i = \Phi_i(t_1, \ldots, t_m, s_1, \ldots, s_m)$, *for* $i = 1, \ldots, r$, *such that*

$$x\Psi_B^{\mathbf{f}}(t_1, \ldots, t_m) = x + \int_0^{t_1} \cdots \int_0^{t_m} x\Psi_B^{\mathbf{f}} \Phi_1^{\#}\Big[\Phi_2^{\#}[\cdots, \cdots], \Phi_3^{\#}[\cdots, \cdots]\Big] ds_1 \cdots ds_m$$

Agrachev-Gamkrelidze formalism

- $x\Phi \stackrel{\text{def}}{=} \Phi(x), \qquad \Phi_1\Phi_2 \stackrel{\text{def}}{=} \Phi_2 \circ \Phi_1.$
- $xf \stackrel{\text{def}}{=} f(x), \qquad \Phi f \stackrel{\text{def}}{=} f \circ \Phi.$
- If $v \in T_xM$, $\quad v\Phi \stackrel{\text{def}}{=} D\Phi(x)(v), \qquad x(f\Phi) \stackrel{\text{def}}{=} D\Phi(x)\,(f(x)).$
- $[f,\, g] \stackrel{\text{def}}{=} fg - gf$ where $\varphi \overset{f}{\mapsto} f\varphi$, $\varphi \overset{g}{\mapsto} g\varphi$ are seen as first order diff. op.s

We have

- $(xf)\varphi = x(f\varphi), \quad (xf)\Phi_1 = x(f\Phi), \quad (x\Phi)f = x(\Phi f),$ $(x\Phi_1)\Phi_2 = x(\Phi_1\Phi_2).$

We denote by

- $(t, x) \mapsto xe^{tf}$ the flow of f.
- If f is complete $\{e^{tf}\}_{t\in R}$ a group of diffeomorphisms on M.

Complicated expression $xfe^{tg}\Phi h\Psi e^{th}k$ makes perfect sense.

AGF – an application

Let $\Pi(t_1,\dots,t_d) \stackrel{\text{def}}{=} e^{t_1 f_1}\cdots e^{t_d f_d}$.
Then $\frac{\partial \Pi}{\partial t_i}(t_1,\dots,t_d) = e^{t_1 f_1}\cdots e^{t_i f_i} f_i e^{t_{i+1} f_{i+1}}\cdots e^{t_d f_d}$,
$\frac{\partial^2 \Pi}{\partial t_i \partial t_j}(t_1,\dots,t_d) = e^{t_1 f_1}\cdots e^{t_i f_i} f_i e^{t_{i+1} f_{i+1}}\cdots e^{t_j f_j} f_j e^{t_{j+1} f_{j+1}}\cdots e^{t_d f_d}$, if $i \le j$.
Thus, $\frac{\partial \Pi}{\partial t_i}(0,\dots,0) = f_i, \qquad \frac{\partial^2 \Pi}{\partial t_i \partial t_j}(0,\dots,0) = f_i f_j$.
Since

$$\sum_{i\le j} f_i f_j t_i t_j + \sum_{i>j} f_i f_j t_i t_j = \sum_{i<j} [f_i, f_j] t_i t_j + \sum_{i=1}^d f_i t_i \Big)^2,$$

by Taylor

$$x e^{t_1 f_1}\cdots e^{t_d f_d} = x + x\sum_{i=1}^d f_i t_i + \frac{1}{2} x \sum_{i<j} [f_i, f_j] t_i t_j + \frac{1}{2} x \Big(\sum_{i=1}^d f_i t_i\Big)^2 + o\Big(\sum_{i=1}^d |t_i|^2\Big)$$

as $(t_1,\dots,t_d) \to (0,\dots,0)$.

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by Taylor

$$x e^{t_1 f_1}\cdots e^{t_d f_d} = x + x\sum_{i=1}^d f_i t_i + \frac{1}{2}x\sum_{i<j}[f_i,f_j]t_i t_j + \frac{1}{2}x\Big(\sum_{i=1}^d f_i t_i\Big)^2 + o\Big(\sum_{i=1}^d |t_i|^2\Big)$$

as $(t_1,\dots,t_d)\to(0,\dots,0)$.
If $d=4$, $f_3=-f_1$, $f_4=-f_2$, $t_3=t_1$, $t_4=t_2$, we (almost) obtain (SAF).

Integral formulas proof sketch

$$\Psi^{\mathbf{f}}_B(t_1,\dots,t_m) = x + \int_0^{t_m} x\Psi^{\mathbf{f}}_B(t_1,\dots,s_m)xV^{\mathbf{f}}_B(t_1,\dots,s_m)ds_m$$

$$\text{if} \qquad xV^{\mathbf{f}}_B(t_1,\dots,t_m) \overset{\text{def}}{=} \frac{\partial}{\partial\tau}\left(x\left(\Psi^{\mathbf{f}}_B(t_1,\dots,t_m)\right)^{-1}\Psi^{\mathbf{f}}_B(t_1,\dots,t_{m-1},\tau)\right)_{\tau=t_m}.$$

One proves

$$V^{\mathbf{f}}_B(\mathbf{t}) = \int_0^{t_{m_1}} \mathrm{Ad}_{\Psi^{\mathbf{f}_{(2)}}_{B_2}(\mathbf{t}_{(2)})}\mathrm{Ad}_{\Psi^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}^{\sigma}_{(1)})}\left(\left[V^{\mathbf{f}_{(1)}}_{B_1}(\mathbf{t}^{\sigma}_{(1)}), V^{\mathbf{f}_{(2)}}_{B_2}(\mathbf{t}_{(2)})\right]\right)d\sigma\,.$$

where $\mathbf{t}\overset{\text{def}}{=}(t_1,\dots t_m)$, $\mathbf{t}_{(1)}\overset{\text{def}}{=}(t_1,\dots,t_{m_1})$, $\mathbf{t}_{(2)}\overset{\text{def}}{=}(t_{m_1+1},\dots,t_m)$
$\mathbf{t}^{\sigma}_{(1)}\overset{\text{def}}{=}(t_1,\dots,t_{m_1-1},\sigma)$, $Ad_\Phi g\overset{\text{def}}{=}\Phi^{\#}g\overset{\text{def}}{=}\Phi g\Phi^{-1}$.
This enables a recursion, and one can conclude. To prove it one shows preliminarily and uses

$$\frac{\partial}{\partial\tau}\left(x\Psi^{\mathbf{f}}_B(t_1,\dots,t_{m-1},\tau)^{-1}\right) = -xV^{\mathbf{f}}_B(t_1,\dots,t_{m-1},\tau)\Psi^{\mathbf{f}}_B(t_1,\dots,t_{m-1},\tau)^{-1}.$$

Chow-Rashevski's theorem

Consider

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x) + u_4 f_4(x) \qquad \text{(CS)}$$

with control constraints $|u_i| \leq 1$, $i = 1, 2, 3, 4$.

Theorem (Chow-Rashevski's theorem (classical))

Let $x_* \in M$. If $f_1, f_2 \in C^2$, $f_3 \in C^1$, $f_4 \in C^0$, and

$$\operatorname{span}\Big\{f_1(x_*), f_2(x_*), f_3(x_*), f_4(x_*), [f_1, f_2], [f_1, f_3](x_*), [f_2, f_3](x_*), [[f_1, f_2], f_3](x_*)\Big\} = T_{x_*}M \qquad \text{(LARC)}$$

Then (CS) is locally controllable from x_* in small time.

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$\exists \varepsilon_0 > 0$, $\exists C \geq 0$ such that $\forall \varepsilon \leq \varepsilon_0$

$$B(x_*, \varepsilon) \subset R(x, \leq C\sqrt[3]{\varepsilon}),$$

where $R(x_*, \leq M)$ is the set of points that can be reached from x_* within time t with $|t| \leq M$ moving along a finite number of arcs of integral lines of vector fields f_i forward or backward in time.

Chow-Rashevski's theorem

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$\exists \varepsilon_0 > 0$, $\exists C \geq 0$ such that $\forall \varepsilon \leq \varepsilon_0$

$$B(x_*, \varepsilon) \subset R(x, \leq C\sqrt[3]{\varepsilon}),$$

where $R(x_*, \leq M)$ is the set of points that can be reached from x_* within time $t \leq M$ by means of a piecewise constant control such that at each time only one of the numbers $u_1(t), u_2(t), u_3(t), u_4(t)$ is nonzero

Chow-Rashevski's theorem

Consider

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x) + u_4 f_4(x) \qquad \text{(CS)}$$

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Then (CS) is locally controllable from x_* in small time.
If (LARC) holds $\forall x_* \in M$, then (CS) is globally controllable.

Slight improvement: (vector fields need to be as many times differentiable as is required for all brackets applied to them to be computable, with continuous highest-order derivatives; in particular, we allow also for just cont. vec. fields).

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$$\dot{x} = u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x) + u_4 f_4(x) \qquad \text{(CS)}$$

with control constraints $|u_i| \leq 1$, $i = 1, 2, 3, 4$.

Theorem (Chow-Rashevski's theorem (classical))

Let $x_* \in M$. If $f_1, f_2 \in C^2$, $f_3 \in C^1$, $f_4 \in C^0$, and

$$\operatorname{span}\Big\{f_1(x_*), f_2(x_*), f_3(x_*), f_4(x_*), [f_1, f_2], [f_1, f_3](x_*), [f_2, f_3](x_*), [[f_1, f_2], f_3](x_*)\Big\} = T_{x_*} M \qquad \text{(LARC)}$$

Then (CS) is locally controllable from x_* in small time.
If (LARC) holds $\forall x_* \in M$, then (CS) is globally controllable.

What about $f_1, f_2 \in C^{1,1}$, $f_3 \in C^{0,1}$?
How to reformulate (LARC) so that it makes sense also for vector fields ~~with this kind of regularity~~?

C.-R.'s Theorem proof sketch

Define

$$\Sigma^{(f_1,f_2)}_{[\cdot,\cdot]}(x,t) \stackrel{\text{def}}{=} \begin{cases} x\Psi^{(f_1,f_2)}_{[\cdot,\cdot]}(\sqrt{t},\sqrt{t})\Psi^{(f_1,f_2)}_{[\cdot,\cdot]}(-\sqrt{t},-\sqrt{t}) & \text{if } t \geq 0 \\ x\Psi^{(f_1,-f_2)}_{[\cdot,\cdot]}(\sqrt{-t},\sqrt{-t})\Psi^{(f_1,-f_2)}_{[\cdot,\cdot]}(-\sqrt{-t},-\sqrt{-t}) & \text{if } t < 0, \end{cases}$$

$$\Sigma^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}(x,t) \stackrel{\text{def}}{=} x\Psi^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}(\sqrt[3]{t},\sqrt[3]{t},\sqrt[3]{t}).$$

Fact. $\Sigma^{(f_1,f_2)}_{[\cdot,\cdot]}$, $\Sigma^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}$ are differentiable at $(x,0)$ with

$$D\Sigma^{(f_1,f_2)}_{[\cdot,\cdot]}(x,0) = (I_n, [f_1,f_2](x)),$$

$$D\Sigma^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}(x,0) = (I_n, [[f_1,f_2],f_3](x)).$$

Even more, these maps are C^1.
Proof not completely trivial, but eventually can be done by asymptotic formulas.

The rest of the theorem's proof is easy: apply **chain rule** and **open mapping** theorem to a map obtained as a suitable composition of maps of the type above.

C.-R.'s Theorem proof sketch (continued)

Take

$$\Theta(t_1,\dots,t_8)\stackrel{\text{def}}{=}x_* e^{t_1 f_1} e^{t_2 f_2} e^{t_3 3_2} e^{t_4 4_2} \Sigma^{(f_1,f_2)}_{[\cdot,\cdot]}(t_5)\Sigma^{(f_1,f_3)}_{[\cdot,\cdot]}(t_6) \Sigma^{(f_2,f_3)}_{[\cdot,\cdot]}(t_7)\Sigma^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}(t_8).$$

By Fact above Θ is C^1 in a nhb. of $(0,\dots,0)\in R^8$, and by **chain rule**

$$D\Theta(0,\dots,0)=\big(f_1(x_*),\ f_2(x_*),\ f_3(x_*),\ f_4(x_*),\ [f_1,f_2](x_*),\ [f_1,f_3](x_*), [f_2,f_3](x_*),\ [[f_1,f_2],f_3](x_*)\big).$$

By (LARC), $D\Theta(0,\dots,0)$ has full rank, so classical **open mapping** theorem yields the result.

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$$\Theta(t_1,\dots,t_8)\stackrel{\text{def}}{=}x_* e^{t_1 f_1} e^{t_2 f_2} e^{t_3 3_2} e^{t_4 4_2} \Sigma^{(f_1,f_2)}_{[\cdot,\cdot]}(t_5)\Sigma^{(f_1,f_3)}_{[\cdot,\cdot]}(t_6) \Sigma^{(f_2,f_3)}_{[\cdot,\cdot]}(t_7)\Sigma^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}(t_8).$$

By Fact above Θ is C^1 in a nhb. of $(0,\dots,0)\in R^8$, and by **chain rule**

$$D\Theta(0,\dots,0)=\big(f_1(x_*),\ f_2(x_*),\ f_3(x_*),\ f_4(x_*),\ [f_1,f_2](x_*),\ [f_1,f_3](x_*), [f_2,f_3](x_*),\ [[f_1,f_2],f_3](x_*)\big).$$

By (LARC), $D\Theta(0,\dots,0)$ has full rank, so classical **open mapping** theorem yields the result. To extend result to vector fields with low regularity we need a generalized differential calculus with "good"

- chain rule,
- open mapping theorem.

H. Sussmann's **GDQ** is our answer to this quest.

Set-valued iterated bracket

If $f_1, f_2 \in C^{1,1}$, $f_3 \in C^{0,1}$, we set

$$[[f_1, f_2],\ f_3]_{set}(x) = \overline{co}\Big\{ \lim_{k\to\infty} \left(Df_3(y_k)[f_1, f_2](x_k) - D[f_1, f_2](x_k)f(y_k)\right) \ : \\ \forall\{x_k\} \subset DIFF([f_1, f_2]),\ \forall\{y_k\} \subset DIFF(f_3),\ x_k \xrightarrow{k} x,\ y_k \xrightarrow{k} x\Big\},$$

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Consider a (formal) iterated bracket B of $deg(B) = m$ and $\mathbf{f} = (f_1, \ldots, f_m) \in C^{B-1,1}$. The map

$$x \mapsto B(\mathbf{f})_{class}(x)$$

is well-defined for almost every x, actually, for $x \in DIFF(\mathbf{f})$.
Suppose $B = [B_1, B_2]$, $deg(B_1) = m_1$, $deg(B_2) = m_2$, so that $m_1 = m_2 = m$. Let r, r_1, r_2 be the number of subbrackets of degree ≥ 2 of B, B_1, B_2, respectively. Clearly $r_1 + r_2 + 1 = r$.
Let $\mathbf{f} = (\mathbf{f}_{(1)}, \mathbf{f}_{(2)})$, where $\mathbf{f}_{(1)} = (f_1, \ldots, f_{m_1})$, $\mathbf{f}_{(2)} = (f_{m_1+1}, \ldots, f_m)$.

Set-valued iterated bracket (continued)

We define recursively

$$B(f)^{(h_1,\dots h_r)}(x) = \left[B_1^{(h_1,\dots,h_{r_1})}(\mathbf{f}_{(1)}),\ B_2^{(h_{r_1+1},\dots,h_{r_1+r_2})}(\mathbf{f}_{(2)})\right](x+h_r)$$

whenever it makes sense, if $r_1, r_2 \geq 1$.
Otherwise, if $r = 1$, which implies $m = 2$, $\mathbf{f} = (f_1, f_2)$, we set
$B(\mathbf{f})^{h_1}(x) = [f_1, f_2]^{h_1}(x) = [f_1, f_2](x + h_1)$.
N.B. Maps $x \mapsto B_1^{(h_1,\dots,h_{r_1})}(\mathbf{f}_{(1)})(x)$, $x \mapsto B_2^{(h_{r_1+1},\dots,h_{r_1+r_2})}(\mathbf{f}_{(2)})(x)$ are only Lipschitz in general, so last def. makes sense only for almost every h_r.

Set-valued iterated bracket (continued)

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We define $B(\mathbf{f})(x)_{set}$ to be the convex hull of limits

$$\lim_{j\to\infty} B(f)^{(h_1^j,\dots h_r^j)}(x)$$

for all $\left\{(h_1^j,\dots h_r^j)\right\}_{j=1}^{\infty}$ of $(\boldsymbol{R}^n)^r$ s. t. $h_i^j \to 0$ as $j \to \infty$ for $i = 1,\dots r$, and ...
Fact. The set-valued map $x \mapsto B(\mathbf{f})(x)$ is upper semicontiuous with compact, convex, nonempty values, and reduces to $\{B(\mathbf{f})_{class}(x)\}$ at $x \in DIFF(\mathbf{f})$.

Comparison with other candidates

Facts taken from F. Rampazzo and H. Sussmann (2007).

Recall
$$[f_1, f_2](x) = Df_2(x) \cdot f_1(x) - Df_1(x) \cdot f_2(x) \qquad \text{(LB)}$$

Candidates:

(i) $[f_1, f_2]_C(x) = \{B \cdot f_1(x) - A \cdot f_2(x) \ : \ A \in D_C f_1(x),\ B \in D_C f_2(x).$

(ii) $[f_1, f_2]_{RS}(x) = [f_1, f_2]_{set}(x)$
$$= \{B \cdot f_1(x) - A \cdot f_2(x) \ : \ (A, B) \in D_C(f_1, g_2)(x)\}.$$

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(i) is "too big"; if $f_1(x) = 1 + |x|$, $[f_1, f_1]_C(0) = [-2, 2]$, while $[f_1, f_1]_{RS}(0) = \{0\}$.

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Question Why don't you set, e.g., for $f_1, f_2 \in C^{1,1}$, $f \in C^{0,1}$

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Question Why don't you set, e.g., for $f_1, f_2 \in C^{1,1}$, $f \in C^{0,1}$

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Answer: Probably, because it is "too small". Natural asymptotic formula not valid in general:

$$\operatorname{dist}\left(x\Psi^{(f_1,f_2,f_3)}_{[[\cdot,\cdot],\cdot]}(t_1, t_2, t_3) - x,\ t_1 t_2 t_3 [[f_1, f_2], f_3]_{RS}(x)\right) = o(t_1 t_2 t_3),$$

as $(t_1, t_2, t_3) \to (0, 0, 0)$.

Theorem (A nonsmooth Chow-Rashevski's theorem)

Let $f_1, f_2 \in C^{1,1}$, $f_3 \in C^{0,1}$, $f_4 \in C^0$, and $x_* \in M$. Consider

$$\dot{x} = u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x) + u_4 f_4(x) \qquad \text{(CS)}$$

with control constraints $|u_i| \leq 1$, $i = 1, 2, 3, 4$.
Assume that for all $v_1 \in [f_1, f_3](x_*)$, $v_2 \in [f_2, f_3](x_*)$, $v_3 \in [[f_1, f_2], f_3](x_*)$

$$\operatorname{span}\left\{f_1(x_*), f_2(x_*), f_3(x_*), f_4(x_*), [f_1, f_2](x_*), v_1, v_2, v_3\right\} = T_{x_*}M. \qquad \text{(LARC-NS)}$$

Then (CS) is locally controllable from x_* in small time.

A proof sketch is given.

Generalized differential quotients

Let $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$, $x_* \in \mathrm{Dom}(F)$, $y_* \in F(x_*)$, and $\Lambda \subset \mathbb{R}^{q\times p}$, compact
Λ is a **GDQ** of F at x_*, written $\Lambda \in GDQ(F; x_*, y_*; \mathbb{R}^p)$ iff . . . , see papers of H. Sussmann.

Properties:

Theorem (Chain rule)

If $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$, $G : \mathbb{R}^q \rightrightarrows \mathbb{R}^r$, $x_* \in \mathrm{Dom}(F)$, $y_* \in \mathrm{Dom}(G)$, $y_* \in F(x_*)$, $z_* \in G(y_*)$, $\Lambda \in GDQ(F; x_*, y_*; \mathbb{R}^p)$, $M \in GDQ(G; y_*, z_*; \mathbb{R}^q)$, *then* $M \circ \Lambda \in GDQ(G \circ F; x_*, z_*; \mathbb{R}^p)$.

Theorem (Open mapping)

Let $F : \mathbb{R}^p \rightrightarrows \mathbb{R}^q$, $x_* \in \mathrm{Dom}(F)$, $y_* \in F(x_*)$, *and* $\Lambda \in GDQ(F; x_*, y_*; \mathbb{R}^p)$. *Suppose that* $\forall L \in \Lambda$, L *is surjective (or* $\mathrm{rank}(L) = q$*). Then, there exists* $\varepsilon :]0, \infty[\ni \delta \mapsto \varepsilon(\delta) \in]0, \infty[$, *linear for small enough* δ *such that* $B_{\varepsilon(\delta)}(y_*) \subset F(B_\delta(x_*))$.

GDQ, a sufficient condition

Proposition (A suffcient condition)

Let $F : \mathbf{R}^p \rightrightarrows \mathbf{R}^q$, $x_* \in U \subset \mathrm{Dom}(F)$, $y_* \in F(x_*)$,, U compact nbh of x_*, $F \lceil U$ with compact graph, $\Lambda \subset \mathbf{R}^{q\times p}$ compact. Suppose that

$$\mathrm{dist}\,(y_* - x_*, \Lambda(x - x_*)) \leq \|x - x_*\|\omega(\|x - x_*\|)$$

for some modulus ω, that is, $\omega(0+) = 0$, and that $\exists\{F^j : U \to \mathbf{R}^q\}_{j=1}^{\infty}$ of single-valued maps s.t.

- $F^j \overset{igc}{\to} F \lceil U$, *that is,* $\sup_{x\in U} \mathrm{dist}\,((x, F_j(x)), \mathrm{Gr}(F)) \to 0$;
- *(equidifferentiability) each F^j is differentiable at x_* and*

$$\|F^j(x) - F^j(x_*) - DF^j(x_*)(x - x_*)\| \leq \|x - x_*\|\omega^j(\|x - x_*\|),$$

  with $\omega^j \to \omega$;

- *and cluster points of $\{DF^j(x_*)\}_{j=1}^{\infty}$ belong to Λ.*

Then $\Lambda \in GDQ(F; x_*, y_*; \mathbf{R}^p)$. bgWave

GDQs of bracket generating multiflows

Let $B = [B_1, B_2]$, $deg(B) = m \geq 1$, $\mathbf{f} = (f_1, \ldots, f_m) \in C^{B-1,1}$. Define, whenever it makes sense,

$$\Sigma^{\mathbf{f}}_B(x,t) \stackrel{\text{def}}{=} x\Psi^{\mathbf{f}}_B(\sqrt[m]{t}, \ldots, \sqrt[m]{t}),$$

if m is odd, and, if m is even,

$$\Sigma^{\mathbf{f}}_B(x,t) = \begin{cases} x\Psi^{\mathbf{f}_{(1)}}_{B_1}(\sqrt[m]{\mathbf{t}_{(1)}})\Psi^{\mathbf{f}_{(2)}}_{B_2}(\sqrt[m]{\mathbf{t}_{(2)}})\Psi^{\mathbf{f}_{(1)}}_{B_1}(\sqrt[m]{\mathbf{t}_{(1)}})^{-1}\Psi^{\mathbf{f}_{(2)}}_{B_2}(\sqrt[m]{\mathbf{t}_{(1)}})^{-1} & \text{if } t \geq 0 \\ x\Psi^{\mathbf{f}_{(1)}}_{B_1}(\sqrt[m]{-\mathbf{t}_{(1)}})\Psi^{-\mathbf{f}_{(2)}}_{B_2}(\sqrt[m]{-\mathbf{t}_{(2)}})\Psi^{\mathbf{f}_{(1)}}_{B_1}(\sqrt[m]{-\mathbf{t}_{(1)}})^{-1}\Psi^{-\mathbf{f}_{(2)}}_{B_2}(\sqrt[m]{-\mathbf{t}_{(2)}})^{-1} & \text{if } t < 0 \end{cases}$$

$$\sqrt[m]{\mathbf{t}_{(1)}} \stackrel{\text{def}}{=} (\sqrt[m]{t}, \ldots, \sqrt[m]{t}) \in \boldsymbol{R}^{m_1}, \qquad \sqrt[m]{\mathbf{t}_{(2)}} \stackrel{\text{def}}{=} (\sqrt[m]{-t}, \ldots, \sqrt[m]{-t}) \in \boldsymbol{R}^{m_2},$$

$$\sqrt[m]{-\mathbf{t}_{(1)}} \stackrel{\text{def}}{=} (\sqrt[m]{-t}, \ldots, \sqrt[m]{-t}) \in \boldsymbol{R}^{m_1}, \qquad \sqrt[m]{-\mathbf{t}_{(2)}} \stackrel{\text{def}}{=} (\sqrt[m]{-t}, \ldots, \sqrt[m]{-t}) \in \boldsymbol{R}^{m_2}.$$

Proposition (GDQs of bracket generating multiflows)

If $\mathbf{f} \in C^{B-1,1}$, *and* $x_* \in M$, *then*

$$(I_n, B(\mathbf{f})(x_*)) \stackrel{\text{def}}{=} \{(I_n, v) \ : \ v \in B(\mathbf{f})(x_*)\} \in GDQ(\Sigma^{\mathbf{f}}_B; (x_*, 0), x_*; \boldsymbol{R}^n \times \boldsymbol{R}).$$

Finishing the proof of the nonsmooth version C.-R.'s Theorem

It suffices to prove, for $v = [f_1^\zeta, f_2^\zeta, f_3^\zeta](x_*)$, or $v \in [[f_1, f_2], f_3]_{set}(x_*)$

$$\left\| x\Psi^{(f_1^\zeta, f_2^\zeta, f_3^\zeta)}_{[[\cdot,\cdot],\cdot]}(t_1, t_2, t_3) - x - t_1 t_2 t_3 v \right\| \leq |t_1 t_2 t_3| \omega^\zeta(|t_1| + |t_2| + |t_3| + \|x - x_*\|), \quad \text{(E)}$$

for $\omega^\zeta \to \omega$ as $\zeta \to 0$, for some modulus ω.
Let $\gamma(\rho) \overset{\text{def}}{=} \sup_{\|h_1\|+\|h_2\| \leq \rho} \operatorname{dist}\left([[f_1, f_2], f_3]^{(h_1,h_2)}(x_*),\ [[f_1, f_2], f_3]_{set}(x_*)\right)$.
Since

$$\text{LHS of (E)} \leq \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \left\| x\Psi^\zeta \Phi_1^{\zeta\#}[\Phi_2^{\zeta\#}[f_1^\zeta, f_2^\zeta], f_3^\zeta] - v \right\| ds_1\, ds_2\, ds_3,$$

by $\|x\Phi^\zeta - x\| \leq C\|\mathbf{t}\|$, $\|D\Phi^\zeta - I_n\| \leq C\|\mathbf{t}\|$ for all $\Phi \in \{\Psi, \Phi_1, \Phi_2\}$, and $\|D^2\Phi_2\| \leq C\|\mathbf{t}\|$, where $\mathbf{t} = (t_1, t_2, t_3)$, and $C \geq 0$ stabilizing as $\zeta \to 0$, $\|\mathbf{t}\| \to 0$, we eventually arrive at (E).