Eikonal equations structured on non-smooth Hörmander type vector fields

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Eikonal equation

Interested in

$$\begin{cases} |\nabla_{\mathcal{X}} u| = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{P}$$

where

$$\mathcal{X} = (X_1, \ldots, X_p) \quad \text{vector fields}\,.$$

$$\nabla_{\mathcal{X}} u := \left(X_1 u, \ldots, X_p u\right)$$

$$|\nabla_{\mathcal{X}} u|^2 := \sum_{j=1}^{p} |X_1 u|^2\,.$$

A viscosity solution, interpretation

Reasonable assumptions $\Longrightarrow$ $\exists$ viscosity solution $\mathbf{u}$

If $g = 0, f = 1$,

$$\mathbf{u}(x) = \text{Minimum time to reach } \mathbb{R}^n \setminus \Omega$$
$$\text{by } \mathcal{X} - \text{trajectories}$$

$\mathcal{X}$ – trajectory := concatenation of a finite no. of integral curves of $X_1, \ldots, X_p, -X_1, \ldots, -X_p$.

Theorem (Typical result)

Assume

1. $X_1, \ldots, X_p$ *satisfy Hörmander's condition of step* $\leq k$ *in* Ω
2. *at each point of* $\partial\Omega$ $\exists$ *Lie bracket of degree* $\leq k$ *non-tangential (i.e. transversal) to* $\partial\Omega$.

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Main fact needed here

$$e^{-tY} \circ e^{-tX} \circ e^{tY} \circ e^{tX}(x) = x + t^2[X, Y](x) + o(t^2) \quad \text{as } t \to 0.$$

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- degree 1

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$$[\mathbf{X_i}, \mathbf{X_j}] := \mathbf{X_i X_j} - \mathbf{X_j X_i} \equiv \nabla_{\mathbf{X_j}} \mathbf{X_i} - \nabla_{\mathbf{X_i}} \mathbf{X_j}$$

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$$[[[X_i, X_j], X_k], X_\ell] \quad \ldots \quad [[X_i, X_j], [X_k, X_\ell]]$$

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- et cetera....

Hörmander's condition and a couple of deep theorems

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Assume $X_1, \ldots, X_p$ *satisfy* **H***ormander's* **C***ondition or* **L***ie* **A***lgebra* **R***ank* **C***ondition, that is,*

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Then

1. *(Chow-Rashevski) Any two points can be connected by an $\mathcal{X}$-trajectory: $T(y,x) \leq C|y-x|^{1/k}$*
2. *(Hörmander)*
$$\mathcal{L} = \sum_{j=1}^{p} X_j^2 \quad \textit{is hypoelliptic}$$
3. *(Bony) $\mathcal{L}$ satisfies the strong maximum principle.*

A set-valued-bracket

If X_1, X_2 are $C^{0,1}$,we set

$$[\mathbf{X_1}, \mathbf{X_2}]_{\mathbf{set}}(\mathbf{x}) := \overline{\mathbf{co}} \left\{ \mathbf{v} = \lim_{\mathbf{j}\to\infty} [\mathbf{X_1}, \mathbf{X_2}](\mathbf{x_j}), \right\}$$

where

1. $x_j \in \mathcal{D}iff(X_1) \cap \mathcal{D}iff(X_2)$ for all j,
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Asymptotic formula: As $|t| + |x - x_*| \to 0$,

$$e^{-tX_2} \circ e^{-tX_1} \circ e^{tX_2} \circ e^{tX_1}(x) - x \in t^2[X_1, X_2](x_*) + t^2 o(1)$$

Higher-order set-valued brackets

If X_1, X_2 are $C^{1,1}$ and X_3 is $C^{0,1}$, we set

$$[[\mathbf{X_1}, \mathbf{X_2}], \mathbf{X_3}]_{\mathbf{set}}(\mathbf{x})$$
$$:= \overline{\mathbf{co}} \left\{ \mathbf{v} = \lim_{\mathbf{j}\to\infty} \mathbf{DX_3(x_k)} \cdot [\mathbf{X_1}, \mathbf{X_2}](\mathbf{x_j}) - \mathbf{D}[\mathbf{X_1}, \mathbf{X_2}](\mathbf{x_j}) \cdot \mathbf{X_3(x_k)}, \right\}$$

where

1. $x_j \in \mathcal{D}iff(DX_1) \cap \mathcal{D}iff(DX_2)\ \forall j$, $x_k \in \mathcal{D}iff(X_3)\ \forall k$,
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Properties: Chart-invariant, robust, u.s.c. with comp, conv values

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$$e^{-tX_3} \circ \underbrace{e^{-tX_1} \circ e^{-tX_2} \circ e^{tX_1} \circ e^{tX_2}}_{\Psi^{-1}} \circ e^{tX_3} \circ \underbrace{e^{-tX_2} \circ e^{-tX_1} \circ e^{tX_2} \circ e^{tX_1}}_{\Psi}(x) - x$$
$$\in t^3[X_1,[X_2,X_3]](x_*) + t^3 o(1)\,.$$

Theorem (A generalised Chow theorem)

Assume $\exists$ *iterated brackets* $B_1, \ldots, B_r$, *possibly set-valued, of the vector fields* $X_1, \ldots, X_p$ *s.t. at* x_*

$$\operatorname{span}\left\{v_1, \ldots, v_r\right\} = \mathbb{R}^n \quad \forall v_1 \in B_1, \ldots v_r \in B_r\,. \qquad \text{(GHC)}$$

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Then *every point* x *in a neighborhood of* x_* *is reached by a* $\mathcal{X}$*-trajectory in minimum time*

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If (GHC) *holds at every* $x_* \in \Omega$, *and* Ω *is connected*, **then** *every two points of* Ω *can be connected by a* $\mathcal{X}$*-trajectory and*

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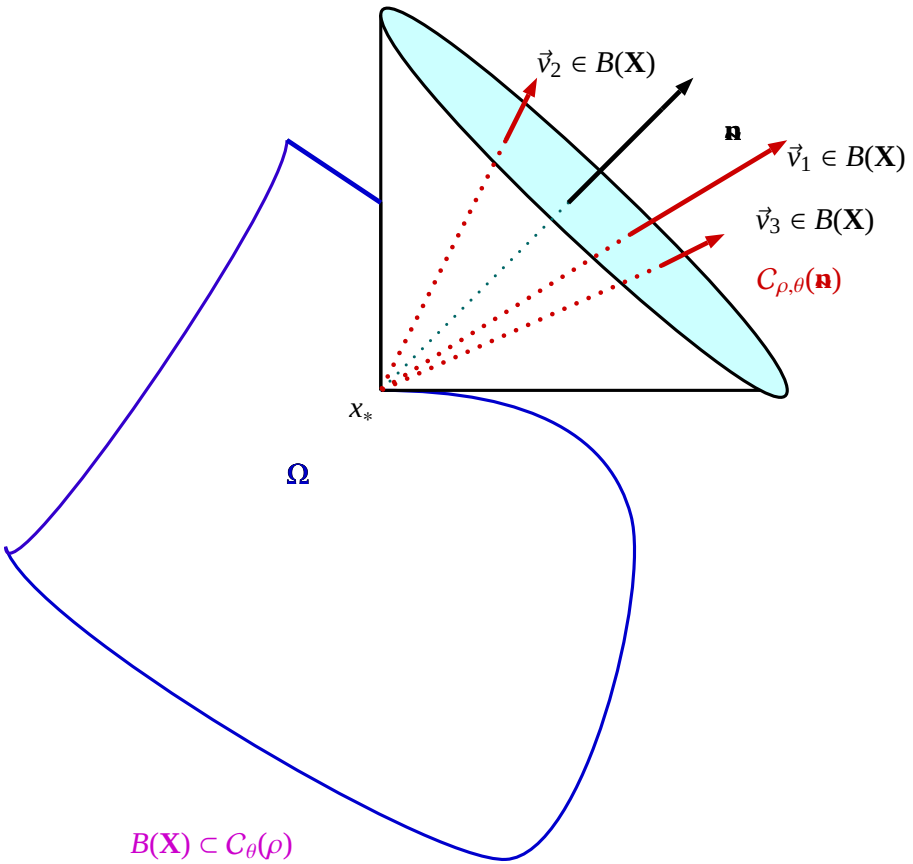
GHC is acronym for **G**eneralized **H**örmander's **C**ondition.
k is step of HGC at x_*.

Transversality condition

$\forall x_* \in \partial\Omega$

$\exists B$, $\deg(B) \leq k$ s.t.

$$B(\mathbf{X})(x_*) \subset C_\theta(\mathbf{n})$$



Theorem

1. *Gen. Hörm cond. of step* $\leq k$ *in* Ω
2. *Transv. cond. of step* $\leq k$ *on* $\partial\Omega$

$\Big\} \Longrightarrow u$ $(1/k)$*–Hölder*

EXAMPLES

Heisenberg group Lie algebra generators
Nonholonomic integrator (Brocket)

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -y \end{pmatrix} \equiv \partial_x - y\partial_z, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x \end{pmatrix} \equiv \partial_y + x\partial_z \,,$$

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Thus a $1/2$-Hölder solution exists for

$$\begin{cases} |\partial_x u - y\partial_z u|^2 + |\partial_y u + x\partial_z u|^2 = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

in every admissible domain.

A modified nonholonomic integrator in dim = 4

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -y \\ 0 \end{pmatrix} \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ x \\ 0 \end{pmatrix} \quad X_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \end{pmatrix}$$

with α a nonvanishing **continuous** function.

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$\Longrightarrow$ 1/2-Hölder minimum time.

Another modification of the nonholonomic integrator

$$X_1 = \begin{pmatrix} 1 \\ 0 \\ -2y + |y| \end{pmatrix} \equiv \partial_x - y\partial_z, \quad X_2 = \begin{pmatrix} 0 \\ 1 \\ 2x + |x| \end{pmatrix} \equiv \partial_y + x\partial_z \,,$$

Simple calculations yield

$$[X_1, X_2] = \left\{ \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \; : \; h \in [2, 6] \right\} \quad \text{for } x = y = 0.$$

In any case **LARC** of step 2 at every point of $\mathbb{R}^3$.
System $\dot{x} = u_1 X_1 + u_2 X_2 + u_3 X_3$, $|u_i| \leq 1$, **controllable**
Minimum time $1/2$-Hölder continuous.

Grushin type vector fields
Higher order brackets

Let

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 0 \\ 2x^k - x|x|^{k-1} \end{pmatrix}$$

One checks (for k even)

$$\underbrace{[X_i, [X_i, [\cdots [X_i, X_{n+i}]]]]_{set}}_{k \text{ bracketings}} = \left\{ \begin{pmatrix} 0 \\ h \end{pmatrix} \; : \; h \in [k!, 3k!] \right\} \quad \text{at } x = 0 \, .$$

Hörmander of step $k+1$.
Hence $1/(k+1)$ minimum time.

Thank you!