Mean Field Games with Hörmander diffusions

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EQUAZIONI DIFFERENZIALI E APPLICAZIONI
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Outline

1. Mean Field Games paradigm and scope of applications
2. References
3. Ergodic games with Hörmander diffusions
4. MFG limit for several populations of agents

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Mean-Field Games

J.-M. Lasry and P.-L. Lions



Goal: Model interactions of large numbers of small similar rational agents.

Tools: PDEs, measure theory, calculus of variations and optimal control, stochastic analysis, game theory, advanced functional analysis, etc.

Let **number of agents** $\mathbf{N} \to \infty$; derive simpler models.

Applications: Economics, Finance, models of social behavior, communication networks etc.

MFG paradigm and applications

MFG = **limits** of **N**-player games as **N** $\to +\infty$.

- **N**-player games with **N large**:
  - At the (**microscopic**) level individuals control their states
  - **Exogenously** specified interaction rules
  - Individuals are rational: they **OPTIMIZE** !!!!
  - Search for **equilibria**: very difficult, NP hard in general.
- If symmetries in the system, interactions can be Mean Field
  - Possible **averaging** effects for large populations
  - **Mean Field limits** easier to study
  - **Macroscopic** behavior of the system through solutions of Mean Field Games

  Lasry - Lions (MFG) Caines - Huang - Malhamé (NCE)

Examples: flocking, schooling, herding, crowd behavior, percolation of information, price formation, hacker behavior and cyber security, ...

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- P. Cardaliaguet, Notes on Mean Field Games (Cardaliaguet's website)
- P.-L. Lions: movies of his courses at College de France
- **Workshops:** Mean Field Games and related topics I–IV (Rome, Padua, Paris, Rome.
- **Monographs** & **books**. Lasry, Lions & Gueant; R. Carmona & F. Delarue; Diogo Gomes et al.; A. Bensoussan et. al
- Other major theoretical **contributors**: Cardaliaguet, Delarue, Porretta, Cirant, Bardi

[1] F. Dragoni & E. Feleqi, Ergodic Mean Field Games with Hörmander diffusions, under review, 2017.

[2] M. Bardi & E. Feleqi, Nonlinear elliptic systems and mean field games, *NoDEA – Nonlinear Differential Equations and Applications*, 2016.

[3] E. Feleqi The derivation of ergodic mean field equations for several populations of players, *Dynamic Games and Applications*, 4, 523–536.

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N-player ergodic stochastic differential games

Minimize ergodic (or long-time average) cost

$$J_i(\alpha^{\mathbf{1}}, \dots, \alpha^{\mathbf{N}}) = \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(L_i(\xi_t^i, \alpha^{\mathbf{i}}{}_t) + F_i(\xi_t^1, \dots, \xi_t^N)\right) dt\right], \quad \text{(Cost)}$$

subject to

$$\begin{cases} d\xi_t^i = \sum_{k=1}^{m_i} b_i^k(\xi_t^i, \alpha_t^i) X_k^i(\xi_t^i) dt + \sum_{k=1}^{m_i} X_k^i(\xi_t^i) \circ dW_t^i \quad \text{in } \mathbb{T}^{d_i} \\ \xi_0^i = x_0^i, \qquad i = 1, \dots. \end{cases} \quad \text{(SDE)}$$

where

- $\alpha^i : [0, \infty[\times\Omega \to A_i$, **controls** (stochastic processes),
- $X_k^i : \mathbb{T}^{d^i} \to \mathbb{R}^{d_i}$, **vector fields**.

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Nash Equilibrium

Definition

$(\hat{\alpha}^1, \ldots, \hat{\alpha}^N)$ is a **Nash equilibrium** if, $\forall i = 1, \ldots, N$,

$$J^i(\hat{\alpha}^1, \ldots, \hat{\alpha}^N) \leq J^i(\hat{\alpha}^1, \ldots, \hat{\alpha}^{i-1}, \alpha^i, \hat{\alpha}^{i+1}, \ldots, \hat{\alpha}^N) \quad \forall \alpha^i.$$

Stationary PDEs

$$\begin{cases} \mathcal{L}_i u_i + \lambda_i + H_i(x, D_{\mathcal{X}_i} u_i) = V^i[m_1, \dots, m_N] \quad \text{in } \mathbb{T}^{d_i}, \\ \qquad\qquad \textit{Hamilton – Jacobi – Bellman} \\ \mathcal{L}_i^* m_i + \operatorname{div}_{\mathcal{X}_i^*}\Big(m_i g_i\Big(x_i, D_{\mathcal{X}_i} u_i\Big)\Big) = 0 \quad \text{in } \mathbb{T}^{d_i}, \\ \qquad\qquad \textit{Kolmogorov – Fokker – Planck} \\ \displaystyle\int_{\mathbb{T}^{d_i}} u_i \, dx^i = 0\,, \quad \int_{\mathbb{T}^{d_i}} m_i \, dx^i = 0, \quad m_i > 0, \end{cases} \qquad i = 1, \dots, N, \tag{1}$$

where

$$\mathcal{L}_i = -\frac{1}{2} \sum_{k=1}^{m_i} (X_k^i)^2 \,.$$

Hörmander's condition:

$$Span\Big(\mathcal{L}\Big(X_1^i(x), \dots, X_{m_i}^i(x)\Big)\Big) = T_x \mathbb{T}^{d_i} \equiv \mathbb{R}^{d_i}, \quad \forall\, x \in \mathbb{T}^{d_i}.$$

Theorem (PDEs and Nash equilibria)

Assume **HC**, $\exists\, \bar{\alpha}^{\mathbf{i}} : \mathbb{T}^{d_i} \times \mathbb{R}^{m_i} \to A_i$, ***Lip. cont.***, *s.t.* $\forall x \in \mathbb{T}^{d_i}, q \in \mathbb{R}^{m_i}$, $\bar{\alpha}^{\mathbf{i}}(x,q)$ *is* ***max. point*** *of* $A_i \ni a \mapsto -b_i(x,a)\cdot q - L_i(x,\alpha) \in \mathbb{R}$, *and* $H_i(x,q) = -b_i(x,\bar{\alpha}^{\mathbf{i}}(x,q))\cdot q - L_i(x,\bar{\alpha}^{\mathbf{i}}(x,q))$ *have* ***linear growth*** *in the gradient* $|H_i(x,q)| \le C(1+|q|)$, *then*

(i) $$\exists \lambda_i \in \mathbb{R},\ u_i \in C^2_{\mathcal{X}_i}(\mathbb{T}^d),\ m_i \in H^1_{\mathcal{X}_i}(\mathbb{T}^{d_i}) \cap L^\infty(\mathbb{T}^{d_i}), \quad i = 1,\dots,N$$
to system (1).

(ii) *Every solution of* (1) *determines a* ***Nash equilibrium*** *in* feedback form *by* $\hat{\alpha}^i(x) = \bar{\alpha}^i\big(x, D_{\mathcal{X}_i} u_i(x)\big)$, $\lambda_i = J_i(\hat{\alpha}^{\mathbf{1}},\dots,\hat{\alpha}^{\mathbf{N}})$ *and* m_i *is the ergodic measure of* $\hat{\xi}^i$, *where* $\hat{\xi} = (\hat{\xi}^1,\dots,\xi^N)$ *is the optimal dynamic* (SDE) *corresponding to the optimal control* $\hat{\alpha} = (\hat{\alpha}^{\mathbf{1}},\dots,\hat{\alpha}^{\mathbf{N}})$.

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Several populations of agents

Assume two populations of N_1 and N_2 agents, resp..
Homogeneous and large populations.
Mathematically, $d_i = d_1$, $m_i = m_1$, $b_k^i = b_k^1$, $X_k^i = X_k^1$, $L^i = L_1$, (hence $\overline{\alpha}^i = \overline{\alpha}_1$) for all $1 \leq i \leq N_1$
and $d_i = d_2$, $m_i = m_2$, $b_k^i = b_k^2$, $X_k^i = X_k^2$, $L^i = L_2$, (hence $\overline{\alpha}^i = \overline{\alpha}_2$) for all $N_1 + 1 \leq i \leq N$.
As a consequence $\mathcal{L}^i = \mathcal{L}_1$, $H^i = H_1$, $g^i = g_1$ for all $1 \leq i \leq N_1$ and $\mathcal{L}^i = \mathcal{L}_2$, $H^i = H_2$, $g^i = g_2$ for all $N_1 + 1 \leq i \leq N$.
For the cost criteria we assume

$$F^i(x^i, \ldots, x^N) = V_1\left[\frac{1}{N_1 - 1} \sum_{\substack{1 \leq j \leq N_1 \\ j \neq i}} \delta_{x^j}, \frac{1}{N_2} \sum_{j=N_1+1}^{N} \delta_{x^j}\right](x^i) \text{ for } 1 \leq i \leq N_1,$$

$$F^i(x^i, \ldots, x^N) = V_2\left[\frac{1}{N_1} \sum_{j=1}^{N_1} \delta_{x^j}, \frac{1}{N_2 - 1} \sum_{\substack{N_1+1 \leq j \leq N \\ i \neq i}} \delta_{x^j}\right](x^i) \text{ for } N_1 + 1 \leq i \leq N,$$

where $V_i : \Big(P(T^d) \times P(\mathbb{T}^d)\Big) \cap \Big(C(\mathbb{T}^d) \times C(\mathbb{T}^d)\Big) \to C^0(\mathbb{T}^d), \quad i = 1, 2$ are continuous operators.

Technical assumption: The ranges of V_i, for $i = 1, 2$, are bounded in $W^{1,\infty}(\mathbb{T}^d)$.

Theorem (symmetrization of Nash equilibria)

All Nash equilibria $\lambda_1^N, \dots, \lambda_N^N \in \mathbb{R}$, $v_1^N, \dots, v_N^N \in C^2(\mathbb{T}^d)$, $m_1^N, \dots, m_N^N \in W^{1,p}(\mathbb{T}^d)$ *satisfy the following properties:*

(i) $\left\{(\lambda_i^N, v_i^N, m_I^N)\right\}_{i,N}$ *is relatively compact in* $\mathbb{R} \times C^2_{\mathcal{X}}(\mathbb{T}^d) \times H^1_{\mathcal{X}}(\mathbb{T}^d)$

(ii)

$$\sup_{1 \le i,j \le N_1} (|\lambda_i^N - \lambda_j^N| + \|v_i^N - v_j^N\|_\infty + \|m_i^N - m_j^N\|_\infty) \to 0 \quad \text{as } N_1 \to \infty,$$

$$\sup_{N_1+1 \le i,j \le N} (|\lambda_i^N - \lambda_j^N| + \|v_i^N - v_j^N\|_\infty + \|m_i^N - m_j^N\|_\infty) \to 0 \quad \text{as } N_2 \to \infty;$$

Theorem (MFG PDEs for two population of agents)

(iii) Let (λ_1, v_1, m_1) *be a limit point of* $\left\{(\lambda_i^N, v_i^N, m_i^N)\right\}_{1\le i\le N_1, N}$ *in* $\mathbb{R}\times C^2(\mathbb{T}^d)\times W^{1,p}(\mathbb{T}^d)$ *and* (λ_2, v_2, m_2) *a limit point of* $\left\{(\lambda_i^N, v_i^N, m_i^N)\right\}_{N_1+1\le i\le N, N}$ *in* $\mathbb{R}\times C^2(\mathbb{T}^d)\times W^{1,p}(\mathbb{T}^d)$. *Then* (λ_i, v_i, m_i), $i=1,2$, *solve the equations*

$$\begin{cases} \mathcal{L}_i v_i + \lambda_i + H_i(x, Dv_i) = V_i[m_1, m_2] & \text{in } \mathbb{T}^d, \\ \mathcal{L}_i^* m_i - div\,(g_i(x, Dv_i)m_i) = 0 & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} v_i\,dx = 0, \quad \int_{\mathbb{T}^d} m_i\,dx = 0, \quad m_i \ge 0 & i=1,2. \end{cases}$$

Proof. One uses the following version of Hewitt and Savage theorem.

$$\lim_{N_1,N_2\to\infty}\int_{\mathbb{T}^{dN_1+N_2}} V\left[\frac{1}{N_1}\sum_{j=1}^{N_1}\delta_{x^j}, \frac{1}{N_2}\sum_{j=N_1+1}^{N}\delta_{x^j}\right]\prod_{j=1}^{N_1} dm_1(x^j)\prod_{j=N_1+1}^{N_1+N_2} dm_2(x^j)$$
$$= V[m_1, m_2] \qquad \forall V\in C(P(\mathbb{T}^d)\times P(\mathbb{T}^d)),\ m_1,\ m_2\in P(\mathbb{T}^d).$$

E. Feleqi, The derivation of ergodic mean field equations for several populations of players, *Dyn. Games and Appl.*, **4** (2013) 523–536.

What's next?

Objective: to extend to more generality the existing theory.

- Existence and regularity of solutions, uniqueness, stability Dynamics described via Hörmader diffusion processes
- Dynamics described via jump-diffusion processes
- Long time behavior and ergodicity
  - $T \to \infty$
- Time periodic solutions
- Rigorous derivation of MFG equations
- The role of boundary condition
- Numerical approximation methods.
- **Problems involving more than one population of agents**.
  - segregation of human populations, Thomas Schelling (Nobel Prize in Economics 2005)